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SYSTEMATIC LOW ORDER CONTROLLER DESIGN FOR DISTURBANCE REJECTION WITH PLANT UNCERTAINTIES

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The research reported herein has concentrated on the development of design methodologies to meet simultaneously several diverse requirements including transient performance, disturbance rejection, robustness, and reliability.

The design methodologies developed herein are based on (1) projective controls, which provide a parametrized family of low-order controllers that guarantee certain performance specifications are met and possess free parameters to be used to meet additional requirements (2) the Frobenius-Hankel (FH) norm as a computationally attractive measure of optimality to meeting disturbance rejection and robustness requirements with low-order projective controllers, and (3) the algebraic Riccati equation as a means of characterizing H-infinity-norm-bounding controllers.

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1 INTRODUCTION

Control of multivariable systems traditionally has been oriented towards meeting requirements of system stability, insensitivity to plant variations, rejection of disturbances, steady state accuracy, and transient performance.

Recently, research has concentrated on the disturbance-rejection problem, and on sensitivity with respect to large plant deviations. The disturbance-rejection problem has been formulated as a worst-case design problem, starting from the classical "game against nature" work, where such a formulation is explicit, through the more recent work on H_∞-norm optimization formulation, where its presence is implicit. Alternatively, minimization of a frequency-weighted H_∞-norm is an appropriate problem formulation for achieving improved robustness of system performance to structured and unstructured plant variations. Frequency-domain-based results on the H_∞ optimal and suboptimal solutions have led, however, to the conclusion that the required controllers are of order higher than the plant. The recent return to the minimax formulations of the disturbance-rejection problem in the time domain has shown that H_∞-norm optimal (and suboptimal) solutions exist in the form of state-feedback and if controls are restricted to output-feedback, in the form of a full-order (same as the plant) observer with modified plant matrix. A complete set of the necessary and sufficient conditions for H_∞-norm controllers expressed via the appropriate Algebraic Riccati Equations (AREs) appears in [1] while earlier work on the connection between the ARE and the H_∞ norm optimization problem leading to these results can be found in [2]. The recent paper [3], reviews the roots and history of the worst case, i.e., minmax, approach to disturbance rejection. These results have shown that the rich theory on the structure and properties of the solutions to the ARE, coupled with the properties of the related Riccati operator and the Riccati inequality, provide extremely useful and fruitful tools for consideration of the classical problems in design of multivariable systems as well as the consideration of important new problems.

The research reported herein has concentrated on the development of design methodologies to meet simultaneously several diverse requirements including transient performance, disturbance rejection, robustness, and reliability, using the following classes of admissible controllers:

- low-order controllers; the goal here is to satisfy the basic performance, disturbance-rejection, and robustness requirements,

- full-order output-feedback controllers, of the same order as the plant; the goal here is to improve the reliability of the system by designing controllers capable of withstanding outages of sensors and actuators, without loosing stability or increasing the H_∞ norm bound;

- output-feedback controllers for decentralized systems; the goal here is to meet requirements associated with transient performance, disturbance rejection and reliability using a decentralized control structure.

The presentation is organized with respect to the classes of admissible controllers. Section 2 and 3 deal mainly with topics related to design of low order controllers. Sections 4, 5, and 6 deal mainly with topics related to state-feedback and full-order output feedback-controllers.

The design methodologies we have developed are based on:

- Projective controls, which provide a parametrized family of low-order controllers that guarantee certain performances specifications are met and possess free parameters to be used to meet additional requirements.

- The Frobenius-Hankel (FH) norm as a computationally attractive measure of optimality to meeting disturbance rejection and robustness requirements with low-order projective controllers.

- The algebraic Riccati equation based characterization of H_∞-norm-bounding controllers, including

  - state-feedback controllers to provide the reference solution for the projective controllers, and

- full-order output-feedback controllers that meet robustness and reliability requirements, or solve the decentralized control problem.

The details of this research have been presented in the references listed below and in manuscripts now in preparation. In the following we highlight the main contributions.

As indicated, projective controls represent a parametrized class of low-order controllers which provide the means for a systematic two-phase design to achieve diverse design objectives. In [4] a methodology was developed which applied projective controls to disturbance attenuation for large flexible structures and other systems with many degrees of freedom. The two-stage design first identifies and parametrizes all strictly proper controllers of given order that retain the dominant system poles (i.e., dynamics) as defined by state-feedback reference dynamics, and then selects a particular controller by determining the free controller parameters to minimize a measure of disturbance attenuation. The measure utilized is the FH norm, the minimization of which is computationally attractive and also places a bound on the H_∞-norm. Restriction of the controllers to the class of projective controllers fixes the system poles for transient behavior and disturbance attenuation while FH-norm minimization then positions system zeros to enhance disturbance attenuation by low-order strictly proper controllers.

In [5] the two-stage design procedure was extended to design multiple control loops for transient performance and disturbance attenuation using a low-order controller in each loop. The H_∞ optimal state-feedback solution was employed to specify and parametrize all decentralized projective controllers that now create fixed modes at desired locations. Then, using the FH-norm minimization approach, the free parameters in all controllers were determined to place the zeroes and remaining poles to augment disturbance attenuation.

The procedures was further extended in [6] to design decentralized projective controls via the H_∞/FH-norm minimization procedure for the case when the controllers are restricted to be strictly proper.

In [7] the FH-norm approach to disturbance rejection was applied to discrete-time systems. A new computational algorithm to minimize the FH norm for controllers of bounded order was developed based on the use of the (discrete) algebraic Riccati equations which, in the limit, reduce to the Lyapunov equations that characterize the necessary conditions. The success of the algorithm is attributed to the expanded regions of existence of positive definite solutions to the Riccati equations, as opposed to Liapunov equations. A nontrivial 5^{th} order example illustrates not only the convergence rate of the algorithm but also the nature of the reduction of the FH norm, the H_∞ norm, the Trace norm and the Hankel norm at each iteration. Also illustrated are bounds on H_∞ norm in terms of the values of the FH norm:

$$\frac{1}{\sqrt{n}}\|G(s)\|_{FH} \leq \|G(s)\|_\infty \leq 2\sqrt{n}\|G(s)\|_{FH}$$

where n is the order of the closed-loop system. In [8] the above approach and the ARE-based computational algorithm were extended to cover in a unified approach three general classes of design problems: disturbance rejection, tracking, and model reference design.

The recent results enabling the construction of H_∞-norm-bounding controllers via the algebraic Riccati equation has stimulated vigorous research into H_∞ designs, to which we have recently made a number of contributions. Our research has encompassed many issues not treated previously by other researchers. These include the development of better bounds on the H_∞-norm for established ARE-based designs [9], and the study of the properties of the convex Riccati operator

$$R(X) = F^T X + XF + \frac{1}{\gamma^2} XGG^T X + H^T H$$

and the associated algebraic Riccati inequality $R(X) \leq 0$ [10]. These properties were fundamental in rederiving in simple terms the state-feedback and output-feedback H_∞-norm-bounding controllers and extending the procedure to achieve robust stabilization with an H_∞-norm bound in the presence of structured uncertainty [11]. Also a new parametrization of all state-feedback controls and output-feedback controls that that guarantee a specified H_∞-norm bound [12] has been obtained.

In [13], [14] the approach was extended to the design of controllers for decentralized systems. It was shown that a controller of the same order as the system can be developed for each control channel by constructing for each channel an observer in which the controls associated with other channels are replaced by the estimates of these controls, as they are defined by the state-feedback solution to the H_∞-norm-bounding problem, and the disturbance is replaced by the worst disturbance as described by the same state-feedback solution. The observer gains for the controllers are determined by the positive definite solution of a large-dimensional ($n \times r$, where r is the number of control channels) Riccati-like algebraic equation.

The developed design methodology was extended to the problem of design of reliable control systems [15]. This includes the design of control systems that possess the following properties:

- stable controllers, i.e., strongly stable closed-loop system,
- robustness to the loss of a selected subset of measurements, and
- robustness to the loss of a selected subset of control inputs.

The essence of our approach stems from the fact that if $X \geq 0$ satisfies $R(X) + P = 0$, where $P \geq 0$, then $R(X) \leq 0$ and consequently stability and H_∞-norm bound can be guaranteed for the base case, while by judicious choice of P one can guarantee additional properties, such as those mentioned above. In [16] the approach was extended to decentralized control structures, and decentralized full order controllers reliable to loss of specified control channels were developed.

The last topic presented in this report deals with H_∞-norm optimal and H_∞-norm-bounding controls for discrete-time systems. Our contributions include the establishment of a lower bound for the achievable H_∞-norm which complements the known upper bound. We have shown [17] that

$$[\lambda_{\max}(G^T P G)]^{1/2} \leq \gamma_{\min} \leq \gamma$$

where $P \geq 0$ satisfies the Discrete ARE (DARE)

$$P = H^T H + A^T P[I + (BB^T - \frac{1}{\gamma^2} G G^T) P]^{-1} A$$

subject to the convexity condition $\gamma^2 - G^T PG > 0$. A study of the properties of the discrete convex Riccati operator and the derivation of the design equations for the output-feedback H_∞-norm-bounding controllers for discrete systems by utilizing a transformation of the DARE to a Generalized (continuous) algebraic Riccati equation (GARE) are given in [18]. A lower bound on the achievable H_∞-norm using output feedback controls was also established.

The presentation of the material has been organized into five Sections. Sections 2 and 3 deal primarily with results related to the design of low-order controllers, and in particular the FH norm and its utilization in design, and with projective controls as a means of defining a suitable parametrized class of low-order controllers. Section 4 establishes the approach used in developing results for state-feedback control, full-order output-feedback control, and decentralized control. Section 5 presents new results on the design of reliable control systems, for the centralized control problem as well as for decentralized control problems where the problem of reliability with respect to a loss of certain control channels is resolved. Section 6 presents extensions of the methodology. Problems considered include robustness to structured parametric uncertainty in the plant, parametrization of classes of state-feedback and output-feedback controls that guarantee an H_∞-norm bound, and the discrete H_∞-norm optimization problem.

2 LOW-ORDER CONTROLLER DESIGN BASED ON THE FH NORM

2.1 Motivation and Problem Formulation

In this report, we present methodologies for design of controllers to achieve closed-loop performance, disturbance rejection, robustness, and reliability for multivariable time-invariant linear systems. The systems will be represented by state-space models or by transfer functions, as may be appropriate in a particular problem setting. In the remainder of this section we specify the analytical representation, and the basic design problems considered in this section.

We consider systems described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw_o \\ y_c &= Hx \\ y &= Cx + Du + w \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $w_0(t) \in \mathbb{R}^q$ is the disturbance, $y_c(t) \in \mathbb{R}^s$ is the controlled output, $y(t) \in \mathbb{R}^d$ is the measured output, and $w \in \mathbb{R}^r$ is measurement noise. In order to insure that the desired control is not achieved at the expense of excessive use of control energy, the controlled output is typically expanded to include the control vector. Thus we will here consider, in general, the controlled output to be

$$z = \begin{bmatrix} y_c \\ u \end{bmatrix} = \begin{bmatrix} Hx \\ Kx \end{bmatrix} = H_c x. \tag{2.2}$$

Two types of controls will be considered: Static output-feedback controls where the controller is of the form

$$u = Ky, \tag{2.3}$$

a particular case of which is the state-feedback controller if $C = I$, and dynamic output-feedback control, where the controller is of the form

$$\begin{aligned} \dot{\xi} &= A_c\xi + B_c y \\ u &= C_c\xi + D_c y. \end{aligned} \tag{2.4}$$

By introducing the extended system describing the coupled dynamics (2.1) and (2.4), the dynamic output-feedback control problem can be reduced to an equivalent static output-feedback problem

$$\begin{aligned} \dot{x}_e &= \tilde{A}x_e + \tilde{B}\tilde{u} + \tilde{G}w_0 \\ y_{ce} &= \tilde{H}x_e + \tilde{E}\tilde{u} \\ y_e &= \tilde{C}x_e + \tilde{D}w \end{aligned} \tag{2.5}$$

with

$$\tilde{u} = K_e y_e \tag{2.6}$$

where

$$\begin{aligned} x_e &= \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \\ \tilde{H} &= [H \quad 0], \quad \tilde{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad \tilde{E} = [E \quad 0], \end{aligned} \tag{2.7}$$

and with the controller parameters packed into the equivalent gain matrix

$$K_e = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \tag{2.8}$$

We now formulate a standard disturbance-rejection problem via the H_∞-norm: Given the closed-loop system

$$\begin{aligned} \dot{x} &= Fx + Gw \\ z &= Hx \end{aligned} \tag{2.9}$$

where w is a disturbance input, z is a system output to be regulated, and the system matrix F depends on the controller parameters, find a controller $K(s)$ which guarantees closed-loop stability and satisfies

$$K(s) = \arg \inf_{K(s)} \|T(K)\|_\infty \tag{2.10}$$

with $T(K;s) = H(sI - F)^{-1}G$. The H_∞ norm is defined as

$$\|T\|_\infty = \sup_\omega \sigma_{\max}\{T(j\omega)\}, \tag{2.11}$$

where $\sigma_{\max}\{\cdot\}$ denotes the maximum singular value. The definition (2.11) signifies that the H_∞ norm represents the largest size of a transfer-function matrix on the $j\omega$ axis. (If $T(s)$ is a scalar transfer function, $\|T\|_\infty$ represents the worst-case amplification of a sinusoidal disturbance input.)

An interpretation of the H_∞ norm in linear systems is that it is the worst-case ratio of output energy to disturbance energy:

$$\|T\|_\infty = \sup_{w \in L_2} \frac{\|z\|_2}{\|w\|_2}. \tag{2.12}$$

Thus, an equivalent formulation of the disturbance-rejection problem (2.10) is to find a controller satisfying

$$K(s) = \arg \inf_{K(s)} \sup_{w \in L_2} \frac{\|z\|_2}{\|w\|_2} \tag{2.13}$$

or

$$K(s) = \arg \inf_{K(s)} \sup\{\|z\|_2 : \|w\|_2 \leq m\}, \tag{2.14}$$

which is a "minimax" problem in dynamic game theory. This reformulation really represents a return to the original formulations of global sensitivity problems as zero-sum games between the control and "nature" (see for example [3]). This was explicitly recognized in recent years, and the minimax formulation has since proved to be the proper vehicle for the characterization and computation of H_∞-norm optimal solutions. It has also been demonstrated that the optimal controllers can be implemented as state-feedback controllers, and that optimal output-feedback controllers are of the same order as the plant.

A related formulation of the disturbance-rejection problem deals with determining suboptimal solutions, which are referred to here as H_∞-norm-bounding controls: Determine K (or K_e) such that the resulting system is stable and

$$\|T\|_\infty < \gamma, \tag{2.15}$$

for selected γ greater than the minimum achievable bound. This formulation has advantages over the H_∞-optimization problem, in that an optimal or near-optimal solution is often characterized by high gains, high sensitivity to design-parameter variations, and excessive concern with the worst disturbance.

For low-order controllers, the H_∞-norm minimization problem and the norm-bounding problem still do not have a computationally tractable solution. This prompts the consideration of alternative formulations. We have developed the FH-norm formulation of the disturbance-rejection problem, and have developed computational algorithms to perform FH-norm optimization. Solutions to the FH-norm optimization problem are easy to compute, and avoid the high-gain and high-sensitivity problems of H_∞-optimal solutions. A relation between the FH norm and the H_∞ norm allows quick determination of an H_∞-norm bound once the FH-norm optimal solution is obtained. The FH-norm approach can be applied to both continuous and discrete systems, and is particularly appealing when the system is linear in the free design parameters. We, therefore, also develop appropriate linear in the free parameter (LIFP) closed-loop systems representations. In Section 3 we proceed to combine the FH-norm approach with the projective controls design methodology.

2.2 The Frobenius-Hankel Norm

Recently, Medanić and Perkins [19] introduced the use of the Frobenius-Hankel (FH) norm, which is defined as the Frobenius norm on the Hankel singular values in disturbance rejection and other control problems. The motivation for the choice of this norm is due to its relationship to more widely known norms such as H_2 and H_∞ and its good computational properties which make it suitable for use in optimization procedures.

In this section, the Frobenius-Hankel norm is defined and its properties explored. In particular, both time-domain and frequency-domain physical interpretations will be given for the FH norm, and a simple computational method will be developed for calculating the FH norm. The FH norm will also be directly related to both the H_2 and H_∞ norms. In the following section, the FH norm will be used as the basis for a parameter-optimization problem and applied to a model-reduction problem and an optimal controller problem.

The Hankel singular values of a stable system are defined as the singular values of the Hankel operator associated with that system. (see [20].) If the system is described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{2.16}$$

with A Hurwitz, then the Hankel singular values σ_i, $i \in \{1, 2, \ldots, n\}$ can be computed as

$$\sigma_i = \lambda_i\{PQ\}, \tag{2.17}$$

where P and Q satisfy

$$AP + PA^T + BB^T = 0 \tag{2.18}$$

$$A^TQ + QA + C^TC = 0. \tag{2.19}$$

Note that P and Q are, respectively, the controllability and observability Grammians of the system, and can be defined by

$$P = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt \tag{2.20}$$

$$Q = \int_0^\infty e^{A^Tt}C^TCe^{At}dt. \tag{2.21}$$

Definition 2.1. *The* **Frobenius-Hankel norm** *of $G(s) \in H_2$ is*

$$\|G(s)\|_{FH} \triangleq \left[\sum_{i=1}^{n}\sigma_i^2\{G(s)\}\right]^{1/2}, \tag{2.22}$$

where $\sigma_i\{\cdot\}$ signifies the i^{th} Hankel singular value.

2.3 Properties of the FH Norm

The FH norm of a given system can be easily computed from its controllability and observability grammians, P and Q.

Theorem 2.1. *Given the system $G(s) \in H_2$ and its controllability and observability grammians, P and Q respectively, then*

$$\|G(s)\|_{FH}^2 = \text{ Tr }\{PQ\}. \tag{2.23}$$

Proof. From Definition 2.1,

$$\|G(s)\|_{FH}^2 = \text{ Tr }\Sigma^2 \tag{2.24}$$

where $\Sigma = \text{diag }(\sigma_1 \ldots \sigma_n)$. Since there exists T nonsingular such that $T^{-T}PT^{-1} = \Sigma$ and $TQT^T = \Sigma$ [20]

$$\|G(s)\|_{FH}^2 = \text{ Tr }(T^{-T}T^T)\Sigma(TT^{-1})\Sigma \tag{2.25}$$

$$= \text{ Tr }(T^T\Sigma T)(T^{-1}\Sigma T^{-T}) \tag{2.26}$$

$$= \text{Tr } PQ \tag{2.27}$$

□

Note that FH-norm computation via Theorem 2.1 involves the solution of the two Lyapunov equations for P and Q, but avoids the eigenvalue computation necessary to determine the individual Hankel singular values.

A time-domain interpretation of the FH norm is as follows:

Theorem 2.2. *Given the system $G(s) \in H_2$ and the impulse response of the system $g(t)$, then*

$$\|G(s)\|_{FH}^2 = Tr \int_0^\infty t\, g(t)^T g(t)\, dt. \tag{2.28}$$

Proof: From Theorem 2.1,

$$\|G(s)\|_{FH}^2 = \text{ Tr } PQ. \tag{2.29}$$

By definitions (2.20) and (2.21), we obtain

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \left[\int_0^T e^{At} BB^T e^{A^T t}\, dt\right] \left[\int_0^T e^{A^T\tau} C^T C e^{A\tau}\, d\tau\right], \tag{2.30}$$

which is equivalent to

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \int_0^T \int_0^T \left[Ce^{A(t+\tau)}B\right] \left[Ce^{A(t+\tau)}B\right]^T dt\, d\tau. \tag{2.31}$$

Let $g(\tau) \triangleq Ce^{A\tau}B$,

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \int_0^T \int_0^T g(t+\tau)^T g(t+\tau)\, dt\, d\tau \tag{2.32}$$

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \int_0^T \int_\tau^{T+\tau} g(t)^T g(t)\, dt\, d\tau. \tag{2.33}$$

Let $H(\tau) \triangleq \int_\tau^{T+\tau} g(t)^T g(t)\, dt$.

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \int_0^T H(\tau)\, d\tau. \tag{2.34}$$

Integrating by parts,

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \left[H(\tau)\tau|_0^T - \int_0^T \tau\, dH(\tau)\right] \tag{2.35}$$

$$\text{Tr } PQ = \lim_{T\to\infty} \text{ Tr } \left[\int_0^T (T-t)g(t+T)^T g(t+T) + tg(t)^T g(t)\, dt\right]. \tag{2.36}$$

In the limit as $T \to \infty$, $g(t+T) \to 0$ and thus

$$\text{Tr } PQ = \text{ Tr } \int_0^\infty tg(t)^T g(t)\, dt. \tag{2.37}$$

□

The following result provides frequency-domain properties of the FH norm.

Theorem 2.3. *Given the system $G(s) \in H_2$ and the frequency response of the system $G(j\omega) = G(s)\mid_{s=j\omega}$, then*

$$\|G(s)\|_{FH}^2 = \frac{j}{2\pi}\int_{-\infty}^{\infty} \text{ Tr } \frac{dG(j\omega)}{d\omega} G(j\omega)^* \, d\omega. \tag{2.38}$$

Proof: Applying Parseval's Theorem to (2.31) yields

$$\|G(s)\|_{FH}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{ Tr } \mathcal{F}[tg(t)]\mathcal{F}[g(t)]^* \, d\omega \tag{2.39}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j \text{ Tr } \left(\frac{dG(j\omega)}{d\omega}\right) G(j\omega)^* \, d\omega \tag{2.40}$$

□

2.4 Relationships with Other Norms

The Frobenius-Hankel norm can be related to the H_∞ norm through the Hankel singular values of the system.

Theorem 2.4. *For a stable system*

$$\bar{\sigma}(G(s)) \leq \|G(s)\|_{FH} \leq \sum_{i=1}^{n} \sigma_i\{G(s)\} \tag{2.41}$$

and

$$\frac{1}{\sqrt{n}}\|G(s)\|_{FH} \leq \|G(s)\|_\infty \leq 2\sqrt{n}\|G(s)\|_{FH}. \tag{2.42}$$

Remark: It has been shown [20], that

$$\bar{\sigma}(G(s)) \leq \|G(s)\|_\infty \leq 2\sum_{i=1}^{n} \sigma_i(G(s)). \tag{2.43}$$

Proof. Consider (2.41). Clearly, $\Sigma_{i=1}^{n}\sigma_i^2 \geq \bar{\sigma}^2$, while

$$\begin{aligned}
\left(\sum_{i=1}^{n}\sigma_i\right)^2 &= \left(\sum_{i=1}^{n}\sigma_i\right)\left(\sum_{j=1}^{n}\sigma_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j \\
&= \sum_{k=1}^{n}\sigma_k^2 + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j \\
&\geq \sum_{k=1}^{n}\sigma_k^2 \\
&\geq \|G(s)\|_{FH}^2.
\end{aligned} \tag{2.44}$$

Consider now (2.42). We have $\sum_{i=1}^{n}\sigma_1^2 \leq n\bar{\sigma}^2$ and so $(1/\sqrt{n})\|G(s)\|_{FH} \leq \bar{\sigma} \leq \|G(s)\|_{\infty}$. On the other hand, we have that given a value of the FH norm,

$$\sigma^T\sigma = \sum_{i=1}^{n}\sigma_{i=1}^2 = \|G(s)\|_{FH}, \quad \sigma^T = [\sigma_1\ \sigma_2 \ldots \sigma_n] \tag{2.45}$$

the maximal value of the sum $\sum_{i=1}^{u}\sigma_i$ is obtained by solving the maximization problem

$$\min\ e^T\sigma, \quad e^T = [1\ 1 \ldots 1], \tag{2.46}$$

subject to (2.45). This leads to the maximizer $\sigma = \lambda e$ where λ must satisfy (2.45), producing $\lambda^2 n = \|G(s)\|_{FH}^2$. But then

$$\sum_{i=1}^{n}\sigma_i = e^T\sigma = \lambda e^T e = \lambda n = \sqrt{n}\|G(s)\|_{FH}. \tag{2.47}$$

For all other values of the σ_i, $\sum_{i=1}^{n}\sigma \leq \sqrt{n}\|G(s)\|_{FH}$ and so

$$\|G(s)\|_{\infty} \leq 2\sum_{i=1}^{n}\sigma_i \leq 2\sqrt{n}\|G(s)\|_{FH}. \tag{2.48}$$

□

The FH norm can also be related to the sensitivity of the H_2 norm to a shift of the eigenvalues of the system along the real axis.

Theorem 2.5. *Let the eigenvalues of the system $G_\alpha(s)$ be given by $\lambda_i = \bar{\lambda}_i + \alpha$, then*

$$\frac{d}{d\alpha}\|G_\alpha(s)\|_2^2\bigg|_{\alpha=0} = 2\|G(s)\|_{FH}^2. \tag{2.49}$$

Proof. The shift in eigenvalues can be expressed by assuming A has the form

$$A(\alpha) = A_o + \alpha I. \tag{2.50}$$

Let

$$J = \|G(s)\|_2^2 = \text{ Tr } PC^TC. \tag{2.51}$$

Then

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = \text{ Tr } P_\alpha C^T C, \tag{2.52}$$

where P_α satisfies

$$AP_\alpha + P_\alpha A^T + 2P = 0. \tag{2.53}$$

Let Q satisfy

$$QA + A^TQ + C^TC = 0 \tag{2.54}$$

then using the properties of the trace, it can be shown that

$$\text{Tr } P_\alpha C^T C = 2 \text{ Tr } PQ. \tag{2.55}$$

Thus

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 2 \text{ Tr } PQ. \tag{2.56}$$

□

This expression may be useful in establishing robustness properties of the system.

2.4.1 Example

To illustrate the relationship between the various norm and the effect of their minimization on the system response consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \\ y_c &= [1 \quad 0]x \\ y &= [0 \quad 1]x \end{aligned}$$

with control restricted to

$$u = -ky = -[0 \quad k]x.$$

It is then easily shown that

$$\|G(s)\|_2^2 = \frac{(1+\alpha^2)k^2+\alpha}{2\alpha k}, \quad k_2^2 = \arg\ \min_k \|G\|_2^2 = \frac{\alpha}{1+\alpha^2}$$

and

$$\|G(s)\|_{FH}^2 = \frac{(1+\alpha^2)k^4+2\alpha^3k^2+2\alpha^2}{4k^2\alpha^2}, \quad k_{FH}^2 = \arg\ \min_k \|G\|_{FH} = \frac{\alpha\sqrt{2}}{\sqrt{\alpha^2+1}}.$$

Moreover,

$$\|G(s)\|_\infty^2 = J(\omega(k),k) \triangleq \max_{\omega^2}\left\{\frac{\omega^2+k^2(1+\alpha^2)}{(\omega^2-\alpha)^2+k^2w^2}\right\}$$

and so

$$\|G(s)\|_\infty^2 = \begin{cases} J(\omega(k),k), & k^2 < \alpha+\alpha\sqrt{\dfrac{\alpha^2+2}{\alpha^2+1}} \\[2ex] \dfrac{k^2(1+\alpha^2)}{\alpha^2} & k^2 \geq \alpha+\alpha\sqrt{\dfrac{\alpha^2+2}{\alpha^2+1}} \end{cases}$$

with $\omega^2(k)$ defined implicitely as satisfying the necessary condition

$$\omega^4+2k^2(1+\alpha^2)\omega^2-\delta=0$$

and

$$\delta = \alpha^2(\alpha^2+2)-(1+\alpha^2)(k^2-\alpha)^2$$

Because for $\delta > 0$, $\|G(s)\|_\infty$ is unimodal and exhibits a maximum, and is monotonically decreasing for $\delta < 0$, it is determined that

$$k_\infty^2 = \arg\ \min_k\ \|G(s)\|_\infty^2 = \frac{1+2\alpha^2-\sqrt{1+\alpha^2}}{\alpha\sqrt{1+\alpha^2}}.$$

The dependency of k_2^2, k_{FH}^2 and k_∞^2 on α is depicted on Figure 2.1.

It is noted that as $\alpha \to \infty$, all gains are bounded, so that high oscillation will result in all cases. The reason is that consideration of the control as an additional controlled output precludes the use of large gains. If the control is not weighted then one obtains

$$\begin{aligned} \|G(s)\|_2^2 &= \frac{k^2+\alpha}{2\alpha k} \\ \|G(s)\|_{FH}^2 &= \frac{k^4+2\alpha^2}{4k^2\alpha^2} \end{aligned}$$

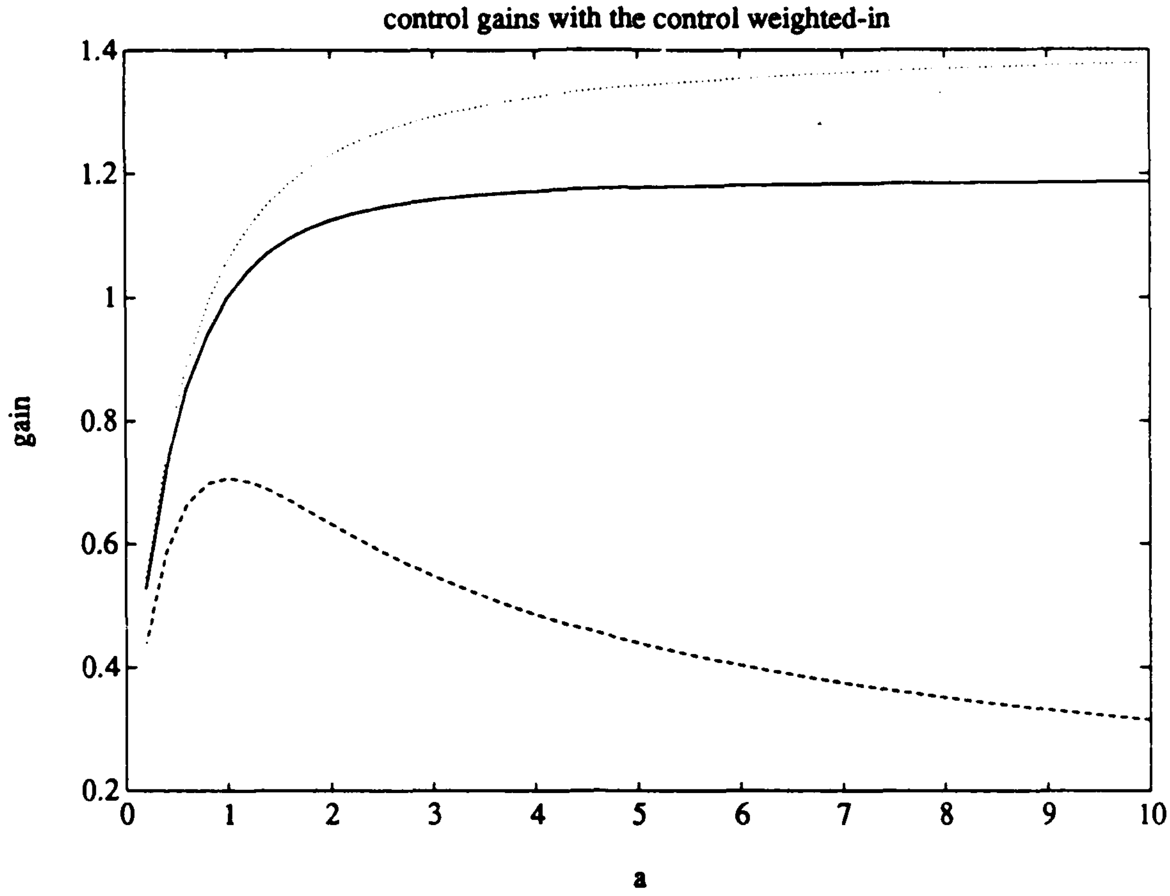

Figure 2.1: Optimal gains.

and

$$\|G(s)\|_\infty^2 = \frac{w^2 + k^2}{(w^2 - \alpha)^2 + k^2 w^2}.$$

From this follows

$$\begin{aligned} k_2^2 &= \alpha \\ k_{FH}^2 &= \alpha\sqrt{2} \\ k_\infty^2 &= \frac{3\alpha}{2} \end{aligned}$$

and the results are displayed in Figure 2.2. Thus, we see again, the characteristic property that the FH norm optimal solution provides more damping than the LQ solution, but not as much as the H_∞-norm optimal solution which, of course, reduces the peak of the gain characteristic. In this case, for all values of α, the H_∞ norm solution guarantees a damping ration $\xi = \frac{\sqrt{3}}{2\sqrt{2}} = 0.6124$, the FH norm solution guarantees $\xi = \frac{\sqrt{2}}{2} = 0.5946$, which the H_2 solution guarantees $\xi = 0.5$.

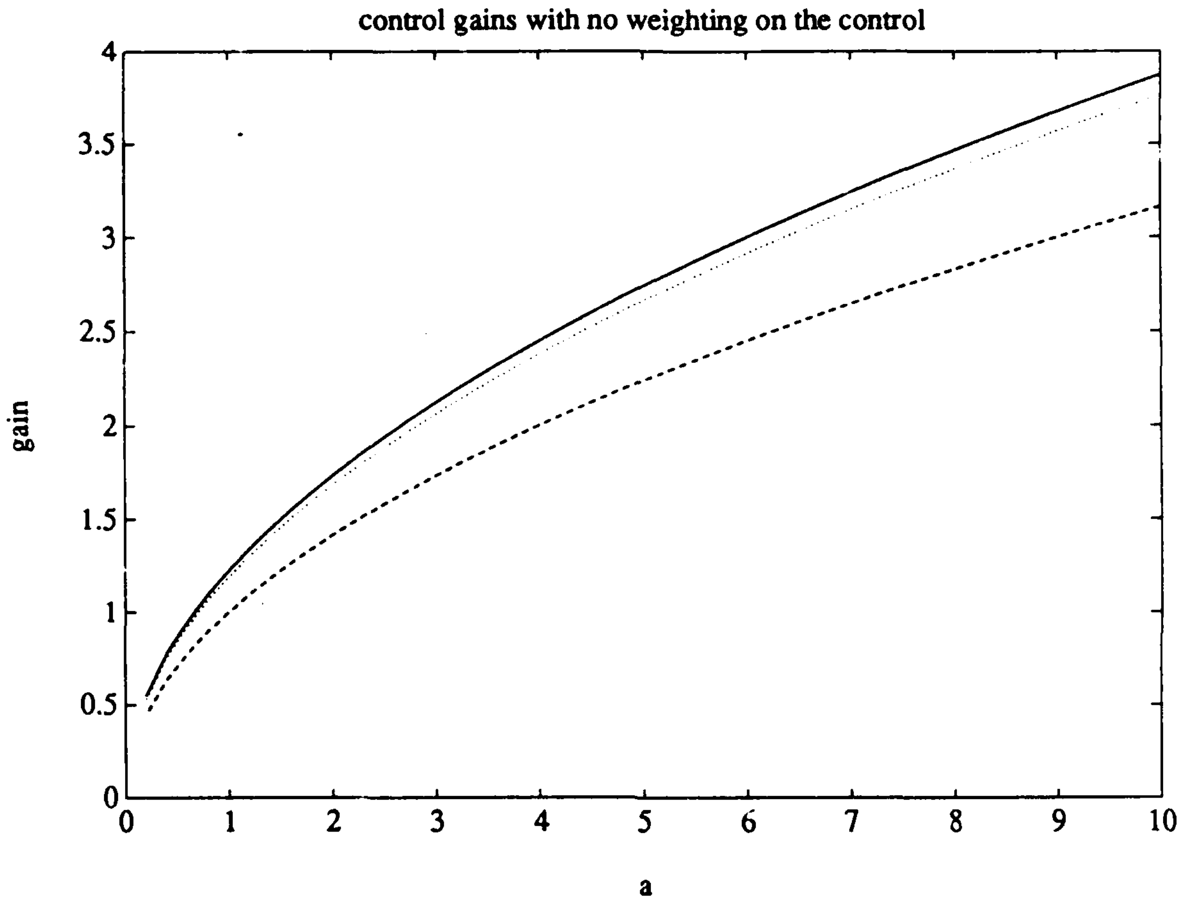

Figure 2.2: Optimal gains.

2.5 Frobenius-Hankel Norm Optimization

In this section, a common framework for solving optimal FH norm problems will be presented. Necessary conditions for an optimal solution will then be formulated and solution methods for solving the necessary conditions will be proposed. In later sections, specific problems will be solved under this framework, and in particular problems related to the design of projective controls.

Let $G(s)$ be a strictly proper system with transfer function

$$G(s) = C(\theta)(sI - A(\theta))^{-1}B(\theta) \tag{2.57}$$

parameterized by θ. The general method used to compute optimal FH norm solutions involves determining the optimal values of the free parameters of the controller. For example, in the case of the model-reduction problem, the parameters represent the reduced-order system, i.e., $\theta = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$. In the controller-synthesis problem, the parameters represent the controller, i.e., $\theta = (A_c, B_c, C_c, D_c)$. The FH norm of $G(s)$ can be computed as

$$J = \|G(s)\|_{FH}^2 = \text{Tr}\ \{PQ\} \tag{2.58}$$

where P and Q satisfy (2.18) and (2.19). The optimization problem is, thus, to find θ such that the criterion (2.58) is minimized subject to the constraints (2.18) and (2.19).

This constrained optimization problem can be converted to an unconstrained optimization problem using Lagrange multipliers. The augmented criterion is given by

$$\hat{J} = \text{Tr}\ \{PQ + M(AP + PA^T + BB^T) + L(A^TQ + QA + C^TC)\}. \tag{2.59}$$

Using this approach, necessary conditions for an optimal solution are

$$\frac{\partial \hat{J}}{\partial P} = A^TM + MA + Q = 0 \tag{2.60}$$

$$\frac{\partial \hat{J}}{\partial Q} = AL + LA^T + P = 0 \tag{2.61}$$

$$\frac{\partial \hat{J}}{\partial L} = AP + PA^T + BB^T = 0 \tag{2.62}$$

$$\frac{\partial \hat{J}}{\partial M} = A^TQ + QA + C^TC = 0 \tag{2.63}$$

$$\frac{\partial \hat{J}}{\partial \theta} = \frac{\partial}{\partial \theta}\ \text{Tr}\ \{2A^T(MP + QL) + B^TMB + C^TCL\} = 0. \tag{2.64}$$

In general, these equations can not be solved for the optimal θ directly. However, iterative methods may be applied to this problem.

A Gradient algorithm approach to the solution of this problem is to find the direction of steepest descent and to take a step in that direction. The direction of steepest descent is in the direction of the gradient with respect to θ; the gradient of J with respect to θ is given by

$$\frac{dJ}{d\theta} = \frac{\partial \hat{J}}{\partial \theta} + \frac{\partial \hat{J}}{\partial P}\frac{d\hat{P}}{d\theta} + \frac{\partial \hat{J}}{\partial Q}\frac{d\hat{Q}}{d\theta} + \frac{\partial \hat{J}}{\partial L}\frac{d\hat{L}}{d\theta} + \frac{\partial \hat{J}}{\partial M}\frac{d\hat{M}}{d\theta}. \tag{2.65}$$

If P, Q, L, M satisfy (2.60-2.63), then

$$\frac{dJ}{d\theta} = \frac{\partial \hat{J}}{\partial \theta}. \tag{2.66}$$

The parameter update is given by

$$\theta_{i+1} = \theta_i - \epsilon \frac{dJ}{d\theta}. \tag{2.67}$$

Basic steps of the algorithm are given in Figure 2.3, for ϵ fixed. This method has been used

1. Select θ_o so that $A(\theta_o)$ is stable.
2. Let $i = 1$.
3. Solve Eqns. (2.60)-(2.63) for L, M, P and Q.
4. Calculate θ_{i+1} from Eqns. (2.66) and (2.67).
5. If the parameters have not converged, let $i = i + 1$ and go to 2.

Figure 2.3: Gradient Algorithm.

in [4] to solve the FH optimization subproblem, associated with the decentralized control of a large space structure using low-order controllers. The steepest decent algorithm results if G is selected so that θ_{i+1} is the minimum of J along the gradient direction.

An alternative approach, referred to here as "the Riccati approach", [21,8] uses Riccati equations instead of Lyapunov equations. The Riccati equations are constructed so that the iterative solution converges to the solution of the Lyapunov equations.

The iterative equations are of the general form

$$\begin{array}{l} A^T M_{i+1} + M_{i+1} A - M_{i+1} R M_{i+1} + M_i R M_i + Q = 0 \\ A L_{i+1} + L_{i+1} A^T - L_{i+1} R L_{i+1} + L_i R L_i + P = 0 \\ A P_{i+1} + P_{i+1} A^T - P_{i+1} R P_{i+1} + P_i R P_i + B B^T = 0 \\ A^T Q_{i+1} + Q_{i+1} A - Q_{i+1} R Q_{i+1} + Q_i R Q_i + C^T C = 0 \end{array} \tag{2.68}$$

where θ_i is the solution of (2.67), or the solution of

$$\frac{\partial \hat{J}}{\partial \theta} = 0, \tag{2.69}$$

if (2.69) can be solved. The second possibility occurs naturally in the discrete case and so the algorithm will be discussed in Section 2.6 in greater detail. Note that if this iterative algorithm converges, it converges to the solution of the corresponding Lyapunov equations. The Riccati approach has the important feature of being solvable for all stabilizable and detectable systems. The Riccati approach should be used when ϵ is fixed, since an unstable plant may then result at any given iteration. However, solving Riccati equations is more time-consuming at each iteration than solving Lyapunov equations; therefore, the increased assurance of convergence is obtained at the cost of greater computational burden.

2.5.1 Optimal model reduction

The disturbance-rejection problem is a useful paradigm for other problems, in particular for the model-reference problem and model-reduction problem. In the model reduction problem, given an n^{th}-order system

$$G(s) = C(sI - A)^{-1}B + D, \tag{2.70}$$

the problem is to find a k-th order approximation

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \tag{2.71}$$

that minimizes $\|G(s) - \hat{G}(s)\|_{FH}$.

The error system as shown in Figure 2.4 is

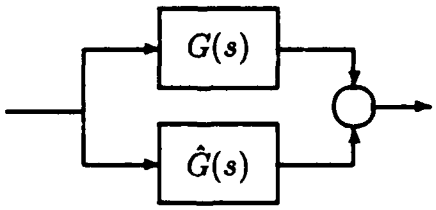

Figure 2.4: Model Reduction Problem.

$$E(s) = G(s) - \hat{G}(s) = C_e(sI - A_e)^{-1}B_e + D_e, \tag{2.72}$$

where

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C_e = \begin{bmatrix} C & -\hat{C} \end{bmatrix}, \quad D_e = D - \hat{D}. \tag{2.73}$$

In order for the error system to be strictly proper, i.e., that $e(t) \to 0$ as $t \to \infty$, we require $D_e = 0$. This is satisfied by letting $\hat{D} = D$.

The necessary conditions from (2.60)–(2.64) are:

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial L} &= A_e P + P A_e^T + B_e B_e^T = 0 \\
\frac{\partial \hat{J}}{\partial M} &= A_e^T Q + Q A_e + C_e^T C_e = 0 \\
\frac{\partial \hat{J}}{\partial P} &= A_e^T M + M A_e + Q = 0 \\
\frac{\partial \hat{J}}{\partial Q} &= A_e L + L A_e^T + P = 0
\end{aligned} \tag{2.74}$$

and

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial \hat{A}} &= 2(MP + QL)_{22} = 0 \\
\frac{\partial \hat{J}}{\partial \hat{B}} &= 2(M_{21} B + M_{22} \hat{B}) = 0 \\
\frac{\partial \hat{J}}{\partial \hat{C}} &= 2(-C L_{12} + \hat{C} M_{2L}) = 0.
\end{aligned} \tag{2.75}$$

The gradient steepest descent or Riccati algorithm can now be used essentially as described.

2.5.2 Disturbance rejection

Given the plant (2.1) controlled by the dynamic controller (2.4), the closed-loop system reduces to (2.5)-(2.7), i.e.,

$$\begin{aligned}
\tilde{A} &= \hat{A} + \hat{B}\hat{K}\hat{C} \\
\tilde{B} &= \hat{G} + \hat{B}\hat{K}\hat{D} \\
\tilde{C} &= \hat{H} + \hat{E}\hat{K}\hat{C} \\
\tilde{D} &= \hat{E}\hat{K}\hat{D}
\end{aligned} \tag{2.76}$$

with

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}, \tag{2.77}$$

$$\hat{E} = [E \;\; 0], \quad \hat{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}, \quad \hat{H} = [H \;\; 0], \quad \hat{K} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \tag{2.78}$$

In order for the closed-loop system to be strictly proper, we require $\tilde{D} = 0$. Thus one restriction on the optimal solution is

$$\hat{E}\hat{D}_c\hat{D} = 0. \tag{2.79}$$

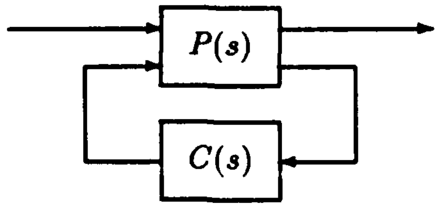

Figure 2.5: Plant with controller configuration for disturbance rejection.

The necessary conditions for an optimal solution are then

$$\begin{aligned}\frac{\partial \hat{J}}{\partial L} &= \tilde{A}P + P\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \\ \frac{\partial \hat{J}}{\partial M} &= \tilde{A}^T Q + Q\tilde{A} + \tilde{C}^T\tilde{C} = 0 \\ \frac{\partial \hat{J}}{\partial P} &= \tilde{A}^T M + M\tilde{A} + Q = 0 \\ \frac{\partial \hat{J}}{\partial Q} &= \tilde{A}L + L\tilde{A}^T + P = 0\end{aligned} \tag{2.80}$$

and

$$\frac{\partial \hat{J}}{\partial \hat{K}} = 2[\hat{B}^T(MP + QL)\hat{C}^T + \hat{B}^T M(\hat{G} + \hat{B}\hat{K}\hat{D})\hat{D}^T + \hat{E}^T(\hat{H} + \hat{E}\hat{K}\hat{C})L\hat{C}^T]. \tag{2.81}$$

Satisfying the condition $ED_cD = 0$ leads to three special cases:

(i) Strictly Proper Controller ($D_c = 0$)

(ii) Noise-Free Measurements ($D = 0$)

(iii) Cheap Control ($E = 0$)

Of course (2.79) can be satisfied in a combination of these three cases. This would imply that some channels of the control would be strictly proper, some noise-free, and some with cheap control.

If the measurement noise (DD^T) is non-singular and the controlled outputs include a non-singular control term ($E^T E$), then the optimal controller must be strictly proper. This case is handled by setting $D_c = 0$ and removing it from the set of parameters to be optimized.

Thus the necessary conditions for an optimal controller are (2.80) and

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial A_c} &= 2(MP + QL)_{22} = 0 \\
\frac{\partial \hat{J}}{\partial B_c} &= 2[(MP + QL)_{21}C^T + M_{22}B_cDD^T] = 0 \\
\frac{\partial \hat{J}}{\partial C_c} &= 2[B^T(MP + QL)_{12} + E^TEC_cL_{22}] = 0.
\end{aligned} \tag{2.82}$$

If the measurement outputs of the plant are noise-free, then $D = 0$ and the necessary conditions for an optimal controller are (2.80) and

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial A_c} &= 2(MP + QL)_{22} = 0 \\
\frac{\partial \hat{J}}{\partial B_c} &= 2[(MP + QL)_{21}C^T + (M_{22}B_c + M_{21}BD_c)DD^T] = 0 \\
\frac{\partial \hat{J}}{\partial C_c} &= 2B^T(MP + QL)_{12} = 0 \\
\frac{\partial \hat{J}}{\partial D_c} &= 2[B^T(M_{11}BD_c + M_{12}B_c)DD^T + B^T(MP + QL)_{11}C^T] = 0.
\end{aligned} \tag{2.83}$$

If, in addition the controller is non-dynamic, i.e., $C(s) = D_c$, then the closed-loop system is

$$G(s) = (H + ED_cC)(sI - A - BD_cC)^{-1}G. \tag{2.84}$$

The necessary conditions for an optimal control are (2.80) and

$$\frac{\partial \hat{J}}{\partial D_c} = B^T(LP + QM)C^T + D_cCMC^T = 0. \tag{2.85}$$

2.5.3 Example

Given the plant

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|cc}
-0.4335 & -0.0118 & -0.9231 & -0.4643 & 0.8854 & -0.7382 \\
-0.9160 & -0.5185 & -0.4110 & -0.0779 & 0.1747 & 1.5473 \\
-0.0414 & -0.6085 & -0.7507 & -0.8901 & -1.4939 & 0.8204 \\
-0.4828 & -0.0916 & -0.2014 & -0.9215 & -1.1423 & -1.5361 \\
\hline
0.9782 & 1.9938 & -0.8140 & -0.8819 & 0 & -1.5443 \\
0.1821 & 0.3387 & 1.6250 & 1.0326 & 0 & 0
\end{array}\right], \tag{2.86}$$

determine a second-order, proper, stabilizing controller $K(s)$ which minimizes the FH norm of the closed-loop system.

To start the algorithm, an initial stabilizing controller is needed. Such a controller is given by

$$K(s) = \left[\begin{array}{cc|c} -1 & 0 & -0.0118 \\ 0 & -1 & -0.0555 \\ \hline 0.0846 & -0.1728 & 0 \end{array}\right]. \tag{2.87}$$

The optimal controller was computed by implementing the steepest decent algorithm using Matlab. Figure 2.6 shows the FH norm ($G = B$, $H = C$) at each iteration of the

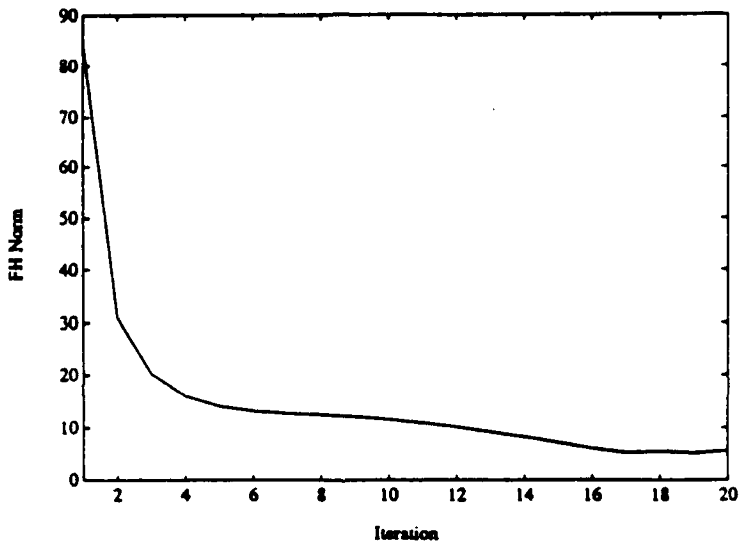

Figure 2.6: Iteration history of FH Norm of System.

algorithm. The FH norm was reduced from its initial value of 84.9 down to 5.6. The optimal controller is determined to be

$$K(s) = \left[\begin{array}{cc|c} -0.7966 & -0.2337 & -0.4563 \\ -0.2502 & -0.7149 & 0.5331 \\ \hline 0.4515 & -0.5542 & 0.1822 \end{array}\right].$$

2.6 Discrete-Time Systems

In this section, we describe in detail the FH-norm approach to disturbance minimization in discrete-time systems, and propose the new Riccati equation based computational algorithm for the design of an FH-optimal controller of selected order.

The formulation, as will be seen, reduces to a linear-in-the-free-parameters (LIFP) system description coupled with a performance criterion that leads to a Parametric Optimization

(PO) problem. The necessary conditions for optimality are derived, and a fixed-point algorithm involving the iterative solution of Lyapunov equations is suggested by the structure of the necessary conditions. Presently available algorithms require an initially stable system. To resolve this initialization problem and aid convergence, a new algorithm is proposed which involves the iterative solutions of discrete Riccati-equations.

2.6.1 The disturbance-rejection problem in discrete-time systems

Consider the disturbance-rejection problem for the system in Figure 2.7, where u is the

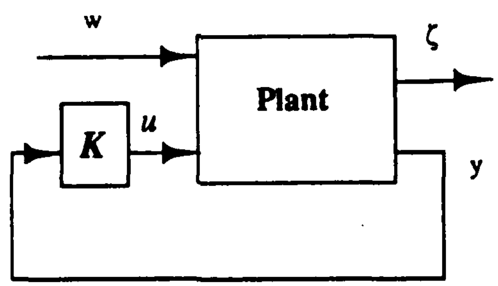

Figure 2.7: System with external disturbance.

control vector, ζ is the controlled output vector, w is the disturbance vector, and y is the measured output vector. The goal is to suppress the response in the output ζ due to the disturbance w. Consider the linear, time-invariant, discrete-time, stochastic state-space system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ew_k \\ y_k &= Cx_k + Fw_k \\ \zeta_k &= Dx_k \end{aligned} \tag{2.88}$$

with $x_k \in \mathbb{R}^n$, $\zeta_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^p$, $y_k \in \mathbb{R}^r$, $w_k \in \mathbb{R}^q$, and $\Sigma_{k=1}^{\infty} w_k^T w_k < \infty$, and with the linear controller

$$\begin{aligned} \xi_{k+1} &= K_1\xi_k + K_2 y_k, \\ u_k &= K_3\xi_k + K_4 y_k \end{aligned} \tag{2.89}$$

where $\xi_k \in \mathbb{R}^s$ and all the parameters of the dynamic compensator are free design parameters. Letting $x_{e,k} = [x_k^T \;\; \xi_k^T]^T$, we get the closed-loop system

$$\begin{aligned} x_{e,k+1} &= A_c x_{e,k} + E_c w_k \\ \zeta_k &= D_c x_{e,k} \end{aligned} \tag{2.90}$$

where

$$A_c = \tilde{A} + \tilde{B}K\tilde{C}, \quad E_c = \tilde{E} + \tilde{B}K\tilde{F}, \quad D_c = [D \;\; 0]\tilde{E} = [E^T \;\; 0]^T, \quad \tilde{F} = [F^T \;\; 0]^T$$

and

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & I_s \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & I_s \end{bmatrix} \quad K = \begin{bmatrix} K_4 & K_3 \\ K_2 & K_1 \end{bmatrix}. \tag{2.91}$$

Then $G_c(z) = D_c(zI - A_c)^{-1}E_c$ represents the closed-loop transfer function from the disturbance input to the regulated output $\zeta(z) = G_c(z)w(z)$.

In the spirit of the H_∞-norm optimization, the optimal solution to the disturbance-rejection problem can be defined as

$$K^\circ = \arg\min_K \|G_c(z)\|_\infty \tag{2.92}$$

where

$$\|G_c(z)\|_\infty \triangleq \max_{|z|=1} \; \sigma_{\max}\{G_c(z)\}. \tag{2.93}$$

Finding a minimum with respect to the H_∞ norm, however, presents computational problems as formidable as in the continuous case since there are no efficient algorithms to solve the ensuing minimax problem involving a controller of constrained structure.

The FH norm of the discrete-time system (2.90), similar to that of a continuous-time system, is defined in terms of Hankel singular values and is computed from the product of the controllability and observability grammians, P and Q. For A_c a stability matrix, P and Q are defined in the discrete case as

$$P \triangleq \sum_{k=0}^{\infty} A_c^k E_c E_c^* (A_c^*)^k, \tag{2.94}$$

$$Q \triangleq \sum_{k=0}^{\infty} (A_c^*)^k D_c^* D_c A_c^*, \tag{2.95}$$

and satisfy the Lyapunov equations

$$\begin{aligned} A_c P A_c^* - P + E_c E_c^* &= 0 \\ A_c^* Q A_c - Q + D_c^* D_c &= 0. \end{aligned} \tag{2.96}$$

The Hankel singular values are defined as

$$\sigma_i(G_c(z)) \triangleq \sqrt{\lambda_i(P\,Q)}, \tag{2.97}$$

and can also be derived from the singular values of the Hankel matrix [20,22]. The FH norm is given by

$$\|G_c(z)\|_{FH} \triangleq \sqrt{\sum_{i=1}^{n+s} \sigma_i^2(G_c(z))} = \sqrt{\sum_{i=1}^{n+s} \lambda_i(PQ)} = \sqrt{\mathrm{Tr}\,(PQ)} \tag{2.98}$$

if $|\lambda_i(A)| < 1 \ \forall i$. Recall that even though the matrices P and Q are not independent of state transformation, the eigenvalues of the product PQ are invariant under such transformations [20].

As has been shown, the FH norm and the Hankel singular values satisfy the bounding relations

$$\sigma_{\max}(G_c(z)) \le \sqrt{\sum_{i=1}^{n+s} \sigma_i^2(G_c(z))} = \|G_c(z)\|_{FH} \le \sum_{i=1}^{n+s} \sigma_i(G_c(z)) \tag{2.99}$$

and

$$\sigma_{\max}(G_c(z)) \le \|G_c(z)\|_\infty \le 2\sum_{i=1}^{n+s} \sigma_i(G_c(z)). \tag{2.100}$$

Introducing the notation for the Trace-norm (T-norm) and Hankel norm

$$\|G_c(z)\|_T = \sum_{i=1}^{n+s} \sigma_i(G_c(z)), \quad \|G_c(z)\|_H = \sigma_{\max}(G_c(z)) \tag{2.101}$$

and the interval $\mathcal{I}$ of the real line defined by

$$\mathcal{I} \triangleq [\|G(z)\|_H, 2\|G_c(z)\|_T], \tag{2.102}$$

it follows that $\|G_c(z)\|_\infty \in \mathcal{I}$ and $\|G_c(z)\|_{FH} \in \mathcal{I}$. Recall also that if $\|G_c(z)\|_{FH} = \delta$, then $\min \|G_c(z)\|_H = \frac{1}{\sqrt{n}}\delta$ while $\max \|G_c(z)\|_T = \sqrt{n}\delta$, and so the largest that $\mathcal{I}$ can be is $[\frac{1}{\sqrt{n}}\delta, 2\sqrt{n}\delta]$, and, similarly, the smallest $\mathcal{I}$ reduces to the set $[\delta, 2\delta]$. Thus, as $\|G_c(z)\|_{FH}$ is reduced by minimization the interval $\mathcal{I}$ is also reduced and the FH norm and the H_∞ norm are forced to move closer together. The FG norm minimization procedure thus provides a near-H_∞-optimal solution, as depicted in Figure 2.8 (where $G_c^*(z)$ is the transfer function of

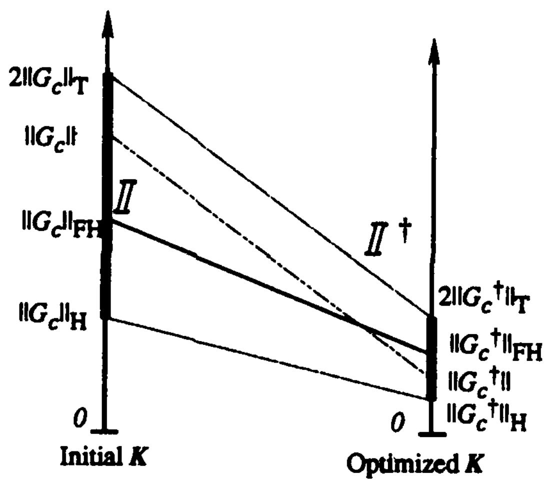

Figure 2.8: The effect of optimization on $\mathcal{I}$.

the optimized system). This will be amply demonstrated by an example in Section 2.7.1.

2.6.2 The FH-norm optimization

The FH-norm optimization in discrete systems problem reduces to the minimization of the criterion

$$J = \|G_c(z)\|_{FH}^2 = \text{tr } (PQ), \tag{2.103}$$

where P and Q satisfy the Lyapunov equation (2.96). The goal of the optimization is to find K^* such that

$$K^* = \arg\min_K \ \mathrm{tr}\ (PQ). \tag{2.104}$$

To convert constrained optimization to an unconstrained optimization, we define again the Lagrange multiplier matrices $L \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n\times n}$, $L = L^T$, and $M = M^T$, following the approach developed for continuous-time case. This leads, in the discrete case, to the extended cost function

$$\hat{J} = \ \mathrm{Tr}\ [PQ + M(A_cPA_c^T - P + E_cE_c^T) + L(A_c^TQA_c - Q + D_c^TD_c)] \tag{2.105}$$

and the following necessary conditions for a minimum:

$$\begin{aligned} \frac{\partial \hat{J}}{\partial P} &= A_c^TMA_c - M + Q = 0 \\ \frac{\partial \hat{J}}{\partial Q} &= A_cLA_c^T - L + P = 0 \\ \frac{\partial \hat{J}}{\partial M} &= A_cPA_c^T - P + E_cE_c^T = 0 \\ \frac{\partial \hat{J}}{\partial L} &= A_c^TQA_c - M + D_c^TD_c = 0 \end{aligned} \tag{2.106}$$

and

$$\frac{\partial \hat{J}}{\partial K} = 2[\tilde{C}(PA_c^TM + LA_c^TQ) + \tilde{F}E_c^TM]\tilde{B} = 0, \tag{2.107}$$

where the derivative of a scalar with respect to a matrix is defined in the usual sense. We see that equations (2.106) are quadratic in the parameter matrix K, and that equation (2.107) is linear in K. Equation (2.107) can be rewritten in the form

$$U_1KV_1 + U_2KV_2 = \Lambda, \tag{2.108}$$

where

$$\begin{aligned} U_1 &= \tilde{B}^TM\tilde{B}, \quad V_1 = \tilde{C}P\tilde{C}^T + \tilde{F}\tilde{F}^T, \quad U_2 = \tilde{B}^TQ\tilde{B}, \quad V_2 = \tilde{C}L\tilde{C}^T \\ \Lambda &= -\tilde{B}^T[(M\tilde{A}P + Q\tilde{A}L)\tilde{C}^T + M\tilde{E}\tilde{F}^T]. \end{aligned}$$

The linear-in-the-parameter form of condition (2.107), which does not arise in the analogous continuous-time problem [23], arises here because of the structure of the discrete Lyapunov equation.

The use of numerical techniques is the only viable approach to the solution of the above necessary conditions. Fixed-point or feasible direction algorithms as suggested by the form of the necessary conditions, may be considered. However, convergence of the feasible direct algorithm is slow while convergence of the fixed point algorithm is not guaranteed: At some iteration, a destabilizing K_j might arise, so that the matrices P_j, Q_j, L_j and M_j will not be positive definite, and the algorithm cannot continue; or, while the algorithm might never encounter this difficulty, it may still not converge. Problems such as this frequently occur in parametric optimization problems (see [24]).

2.7 The Riccati-Based Algorithm

To improve computational efficiency, resolve the initialization problem [24], and achieve convergence, we use an algebraic Riccati-equation approach for computing the FH-norm optimal controller. This approach is the forerunner of the Riccati approach mentioned in Section 2.5 for continuous-time case, and is treated here in greater detail. The use of the Riccati equations is again proposed because of robust properties of positive semi-definite solutions of these equations, and because of their relationship to the corresponding Lyapunov equations (2.106).

Consider the Discrete Algebraic Riccati Equation (DARE)

$$APA^T - P + S - AP(P+R)^{-1}PA^T = 0 \tag{2.109}$$

for $S \geq 0$, $R > 0$. This has the same terms as the Lyapunov equation except for the "rational" term $AP(P+R)^{-1}PA^T$. Recall the following fundamental property.

Lemma 2.1 [25]. *If (A,S) is a stabilizable pair, then there exists a $P > 0$ that solves equation (2.109).* □

If S is positive definite then (A,S) is obviously stabilizable, and so P is positive definite regardless of the stability of A. This fact is exploited to construct an algorithm that overcomes the initial stabilization problem. This property also guarantees the continuation of the algorithm through successive iterations by generating positive definite solutions to Riccati equations constructed from the Lyapunov equations (2.106).

To adapt the Lyapunov-equation type conditions into a Riccati setting, we examine, for example, the j^{th} iterate of the third equation in (2.106)

$$0 = A_{c,j}P_{j+1}A_{c,j}^T - P_{j+1} + S_j \tag{2.110}$$

where $S_j = E_{c,j}E_{c,j}^T$. We may expand this into the DARE form

$$0 = A_{c,j}P_{j+1}A_{c,j}^T - P_{j+1} + S_j' - A_{c,j}P_{j+1}(P_{j+1} + R)^{-1}P_{j+1}A_{c,j}^T \tag{2.111}$$

where S_j' is now given by $S_j' = S_j + A_{c,j}P_j(P_j + R)^{-1}P_jA_{c,j}^T$. If P is a fixed point of this algorithm, (i.e., if $P_j \to P$ as $j \to \infty$), then in the limit (2.111) converges to (2.110). We use the same expansion technique on the other three Lyapunov equations and construct an algorithm based on the iteration of the obtained Riccati equation in the spirit of fixed point algorithms.

Algorithm:

1) Set K_o (arbitrary), $R > 0$, $\epsilon > 0$, and let $P_o = Q_o = L_o = M_o = I$

2) Compute $A_{c,j} = \tilde{A} + \tilde{B}K_j\tilde{C}, E_{c,j} = \tilde{E} + \tilde{B}K_j\tilde{F}$

3) Solve the DARE equations

$$\begin{aligned}
0 &= A_{c,j}P_{j+1}A_{c,j}^T - P_{j+1} + S_{1j} - A_{c,j}P_{j+1}(P_{j+1} + R)^{-1}P_{j+1}A_{c,j}^T \\
0 &= A_{c,j}^TQ_{j+1}A_{c,j} - Q_{j+1} + S_{2j} - A_{c,j}^TQ_{j+1}(Q_{j+1} + R)^{-1}Q_{j+1}A_{c,j} \\
0 &= A_{c,j}L_{j+1}A_{c,j}^T - L_{j+1} + S_{3j} - A_{c,j}L_{j+1}(L_{j+1} + R)^{-1}L_{j+1}A_{c,j}^T \\
0 &= A_{c,j}^TM_{j+1}A_{c,j} - M_{j+1} + S_{4j} - A_{c,j}^TM_{j+1}(M_{j+1} + R)^{-1}M_{j+1}A_{c,j}
\end{aligned}$$

for $P_{j+1}, Q_{j+1}, L_{j+1}$, and M_{j+1}, where

$$\begin{aligned}
S_{1j} &= E_{c,j}E_{c,j}^T + A_{c,j}P_j(P_j + R)^{-1}P_jA_{c,j}^T, \\
S_{2j} &= D_c^TD_c + A_{c,j}^TQ_j(Q_j + R)^{-1}Q_jA_{c,j} \\
S_{3j} &= P_j + A_{c,j}L_j(L_j + R)^{-1}L_jA_{c,j}^T, \\
S_{4j} &= Q_j + A_{c,j}^TM_j(M_j + R)^{-1}M_jA_{c,j}
\end{aligned}$$

4) Solve equation (2.108) for K_{j+1}

5) If $||K_{j+1} - K_j|| \geq \epsilon$, go to 2); else stop.

This approach has several appealing features. First of all, P_j and Q_j (and hence L_j and M_j) always have positive definite solutions if S_{1j} and S_{2j} are positive definite. This will always be true if P_o and Q_o are chosen positive definite (e.g., $P_o = Q_o = I$). The second feature is the introduction of the matrix R in (2.111) which can be used to control speed of convergence of the algorithm. This property can be seen in that for $R > 0$ and large, the Riccati equation (2.111) approaches the corresponding Lyapunov equation (2.110).

If K_j is not stabilizing for some $j \geq 0$, the algorithm still generates positive definite P_j, Q_j, L_j, M_j, and so the updates of K_j may still be obtained uniquely from (2.108). In particular, any $K_o \in \mathbb{R}^{(p+s)\times(r+s)}$, can be used as an initial starting point, (unlike algorithms that iterate on Lyapunov equations), resolving the initialization problem. The Riccati solutions may be obtained by eigenvector or Schur methods [26]. Equation (2.108) may be solved by Schur methods [27].

2.7.1 An example

The example used is a fifth-order plant and second-order controller. There are three measured outputs, two control inputs, two disturbances, and two regulated outputs. The system was open-loop unstable and the initial gain, chosen at random, was not stabilizing. The matrices in this example are

$$A = \begin{bmatrix} 0.7090 & 0.2174 & 0.2156 & 0.2471 & 0.2714 \\ 0.9167 & 0.6322 & 0.4611 & 0.1519 & 0.1583 \\ 0.4492 & 0.0208 & 0.9797 & 0.2248 & 0.3055 \\ 0.2074 & 0.6609 & 0.6527 & 0.4585 & 0.2312 \\ 0.6020 & 0.9731 & 0.5431 & 0.5976 & 0.9319 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4218 & 0.6696 \\ 0.9280 & 0.7232 \\ 0.3669 & 0.7510 \\ 0.2272 & 0.3142 \\ 0.4842 & 0.3740 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5190 & 0.3851 & 0.6518 & 0.1310 & 0.2305 \\ 0.5971 & 0.4729 & 0.5466 & 0.5970 & 0.5064 \\ 0.3805 & 0.3592 & 0.8039 & 0.2023 & 0.1848 \end{bmatrix}, \quad F = \begin{bmatrix} 0.7631 & 0.4555 \\ 0.0501 & 0.5027 \\ 0.6824 & 0.2716 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.1958 & 0.5790 & 0.8710 & 0.9427 & 0.0715 \\ 0.9716 & 0.8839 & 0.7459 & 0.6631 & 0.7721 \end{bmatrix}, \quad E = \begin{bmatrix} 0.6911 & 0.8366 \\ 0.9911 & 0.9238 \\ 0.1412 & 0.9555 \\ 0.7691 & 0.2172 \\ 0.4180 & 0.6685 \end{bmatrix},$$

with the initial gain matrix (partitioned as in (2.91))

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = \begin{bmatrix} 0.9074 & 0.4859 & 0.5117 & 0.5784 & 0.7800 \\ 0.6664 & 0.2697 & 0.7345 & 0.0293 & 0.6229 \\ & & & & \\ 0.4865 & 0.5581 & 0.0614 & 0.8401 & 0.2379 \\ 0.3389 & 0.4706 & 0.3855 & 0.3024 & 0.9400 \end{bmatrix},$$

In simulations it was seen that using equation (2.108) directly to produce the update of K causes the step size between updates to be large regardless of the size of the matrix R. A modification of this algorithm to control the step size was seen to be useful. This may be readily accomplished since the updates of the gain need not be stabilizing. An update law for the gain K was chosen to be

$$K_{j+1} = \alpha N_j + (1-\alpha)K_j, \quad \alpha \in [0,\ 1]$$

where N_j is the solution of equation (2.108) in the j^{th} iteration.

Various choices of the gain α, and the matrix R were analyzed. The variation of the four norms, the H_∞ norm, the FH norm, the Hankel norm, and the trace norm at each iteration, where the values $R = 5I$ and $\alpha = 0.6$ were used, is shown in Figure 2.9. The value of K at iteration 40 was

$$K_{40} = \begin{bmatrix} -15.8530 & 2.5021 & 17.2028 & -3.1335 & -3.9396 \\ 11.2437 & -3.1656 & -13.5988 & 3.1284 & 3.5185 \\ 2.0500 & 0.0137 & -2.5740 & 1.1343 & 0.8562 \\ -2.5727 & -0.0660 & 3.3596 & -1.6122 & -0.8960 \end{bmatrix}.$$

In this example, we can see that beyond 20 iterations the various norms decrease exponentially. In Figure 2.10 we see that the algorithm keeps finding gains that reduce the FH norm, even though the convergence of the gain K is not smooth. The use of a step-size control is seen to be useful in this case. The FH norm and the H_∞ norm, in this example, are seen to be close not only at the optimum but also at each iteration. We can also see that in this example, the solution is stabilizing, and also produces a stable controller.

In this example, the interval $\mathcal{I} = [6.00,\ 25.25]$, bounds the H_∞ norm which was 9.44 at the point in which the first stabilizing gain was determined. After 30 iterations, the interval bounding the H_∞ norm was reduced to [.0105, .0399], and its actual value was .014. Thus,

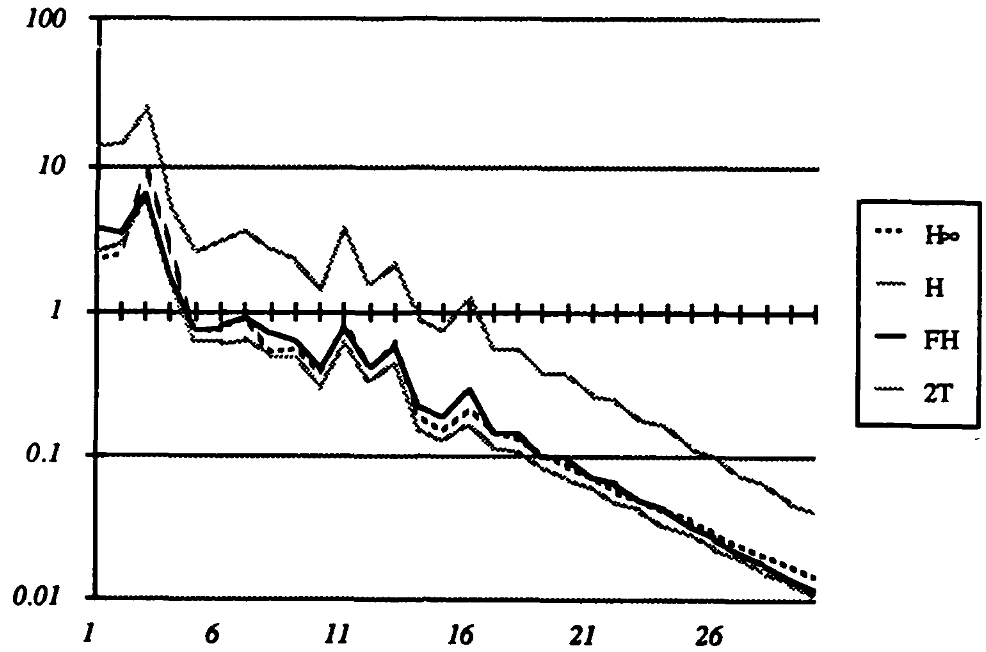

Figure 2.9: Log of various norms at each iteration.

a significant reduction was made in the H_∞ norm and also the bounding interval $\mathcal{I}$. The fact that the interval can be monitored implies that its size and location may be enough to identify the size of the H_∞ norm. Thus, the desired accuracy of the H_∞ norm may be obtained without actual computation.

Also considered was the effect of varying the order of the controller. Figure 2.11 shows the FH norm trajectories in the computation process. As expected, the size of the controller affects the reduction of the H_∞ norm. Each of the initial controllers chosen were unstable and resulted in unstable closed-loop systems. After fifty iterations, with the controller of order $s = 1$, the interval $\mathcal{I}$ was [0.0998, 0.5694], and the FH norm was 0.1448, the H_∞ norm being 0.1254. For the controller of second order, $\mathcal{I}$ was reduced to [.000171, .000583], and the FH norm was .000182, while the H_∞ norm was .000228. Finally, for the third order controller, $\mathcal{I}$ was [1.80e-6, 1.10e-5], and the FH and H_∞ norms were computed to be 4.38e-6 and 2.00e-6 respectively. It appears that the first-order controller converged with an FH

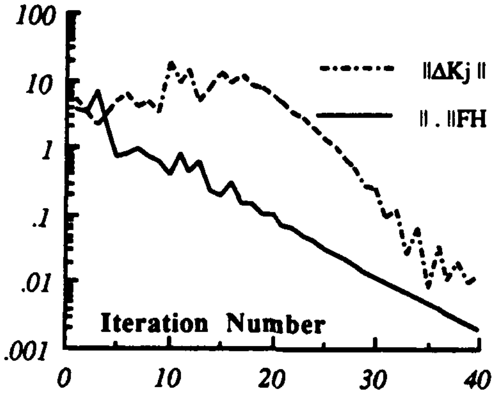

Figure 2.10: Variation of norm of ΔK_j and the FH norm.

norm of approximately .1448, while the algorithm produced controllers of order 2 and 3 with significantly smaller FH norm.

2.7.2 Application to other design problems

The disturbance-rejection paradigm can be used to treat other control problems in a common framework. Prominent examples are the tracking of exogenous inputs, and model reference design [28]. These are together with the disturbance-rejection problem depicted in Figures 2.12. As before, u is the control vector, ζ is the controlled output vector, w is the disturbance vector, and y is the measured output vector, while r is a reference signal. In the tracking problem, we also have the exogeneous input $\hat{w}$. The goal of the disturbance rejection problem is to suppress the response in the output ζ due to the disturbances w. The objective of the tracking problem is to minimize the "throughput" from the input vector $v = [\hat{w}^T \; w^T]^T$ to the output e. Similarly the goal of the model reference problem is to find

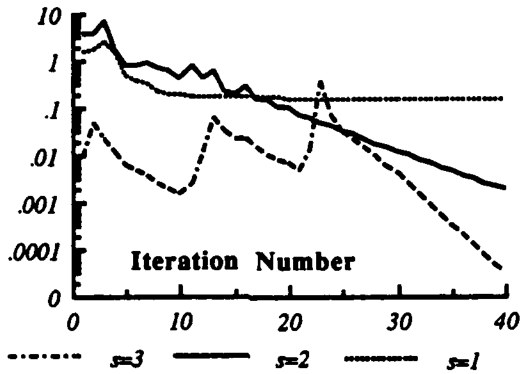

Figure 2.11: FH norm variation for controllers of different order.

a controller that reduces the effect of the "disturbance" input vector $v = [r^T \ w^T]^T$ to the output vector e.

Developing a common design methodology, we first find a common way of representing the closed-loop system in the tracking and model reference design in a linear-in-the-parameters fashion, considering again the linear, time-invariant, discrete-time, stochastic state-space plant (2.88).

We consider additionally in the tracking problem the tracking model

$$\begin{aligned} \mu_{k+1} &= A_1\mu_k + B_1\hat{w}_k \\ r_k &= C_1\mu_k \end{aligned} \tag{2.112}$$

and the controller

$$\xi_{k+1} = K_1\xi_k + [K_2 \ K_3]\begin{bmatrix} y_k \\ r_k \end{bmatrix}, \quad u_k = K_4\xi_k + [K_5 \ K_6]\begin{bmatrix} y_k \\ r_k \end{bmatrix}, \tag{2.113}$$

and so for $x_{e,k} = [\mu_k^T \ x_k^T \ \xi_k^T]^T$, $v_k = [\hat{w}_k^T \ w_k^T]^T$, and $e_k = r_k - \zeta_k$ we have

$$x_{e,k+1} = A_{c,d}x_{e,k} + E_{c,d}v_k, \quad e_k = D_d x_{e,k} \tag{2.114}$$

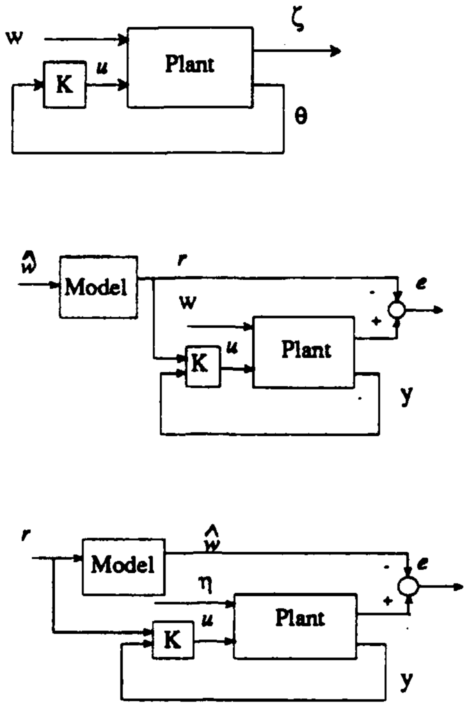

Figure 2.12: System structure for disturbance rejection, model reference and tracking.

where $A_{c,t} = \tilde{A}_t + \tilde{B}_t K_t \tilde{C}_t$, $E_{c,t} = \tilde{E}_t + \tilde{B}_t K_t \tilde{F}_t$, $D_t = [C_1 \;\; -D \;\; 0]$,

$$\tilde{A}_t = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B}_t = \begin{bmatrix} 0 & 0 \\ B & 0 \\ 0 & I_s \end{bmatrix}, \quad \tilde{C}_t = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I_s \end{bmatrix},$$

$$\tilde{E}_t = \begin{bmatrix} B_1 & 0 \\ 0 & E \\ 0 & 0 \end{bmatrix}, \quad \tilde{F}_t = \begin{bmatrix} 0 & 0 \\ 0 & F \\ 0 & 0 \end{bmatrix}, \quad \tilde{K}_t = \begin{bmatrix} K_6 & K_5 & K_4 \\ K_3 & K_2 & K_1 \end{bmatrix}.$$

For the model reference problem, we adjoin to (2.90) the model

$$\begin{aligned} \mu_{k+1} &= A_1 \mu_k + B_1 r_k \\ \hat{w}_k &= C_1 \mu_k \end{aligned} \tag{2.115}$$

and the controller

$$\xi_{k+1} = K_1 \xi_k + [K_2 \; K_3] \begin{bmatrix} y_k \\ r_k \end{bmatrix}, \quad u_k = K_4 \xi_k + [K_5 \; K_6] \begin{bmatrix} y_k \\ r_k \end{bmatrix}, \tag{2.116}$$

and so for $x_{e,k} = [\mu_k^T \; x_k^T \; \xi_k^T]^T$, and $v_k = [r_k^T \; w_k^T]^T$, we have

$$x_{e,k+1} = A_{c,d} x_{e,k} + E_{c,d} v_k, \quad e_k = D_d x_{e,k} \tag{2.117}$$

where $A_{c,m} = \tilde{A}_m + \tilde{B}_m K_m \tilde{C}_m$, $E_{c,m} = \tilde{E}_m + \tilde{B}_m K_m \tilde{F}_m$, $D_m = [C_1 \; -D \; 0]$,

$$
\begin{aligned}
\tilde{A}_m &= \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B}_m = \begin{bmatrix} 0 & 0 \\ B & 0 \\ 0 & I_s \end{bmatrix}, \quad \tilde{C}_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I_s \end{bmatrix}, \\
\tilde{E}_m &= \begin{bmatrix} B_1 & 0 \\ 0 & E \\ 0 & 0 \end{bmatrix}, \quad \tilde{F}_m = \begin{bmatrix} I_p & 0 \\ 0 & F \\ 0 & 0 \end{bmatrix}, \quad \tilde{K}_m = \begin{bmatrix} K_6 & K_5 & K_4 \\ K_3 & K_2 & K_1 \end{bmatrix}.
\end{aligned}
$$

All the parameters $K_1, \ldots, K_6$ of the various dynamic compensators are free design parameters.

For each of these problems,

$$G_c(z) \triangleq D_c(zI - A_c)^{-1}E_c \tag{2.118}$$

represents the closed-loop transfer function from the "disturbance" input to the output, $e(z) = G_c(z)v(z)$. We thus have a common representation for each of the three problems, which reduce to a disturbance rejection problem.

2.7.3 Examples

Using the same fifth-order plant and second order controllers. There are three observations, two control inputs, two disturbances, and two outputs. The system was open-loop unstable and the initial gains, for both the model reference and the tracking problems, were chosen at random and were not stabilizing. The data defining the examples are

$$A = \begin{bmatrix} 0.7090 & 0.2174 & 0.2156 & 0.2471 & 0.2714 \\ 0.9167 & 0.6322 & 0.4611 & 0.1519 & 0.1583 \\ 0.4492 & 0.0208 & 0.9797 & 0.2248 & 0.3055 \\ 0.2074 & 0.6609 & 0.6527 & 0.4585 & 0.2312 \\ 0.6020 & 0.0931 & 0.5431 & 0.5976 & 0.9319 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4218 & 0.6696 \\ 0.9280 & 0.7232 \\ 0.3669 & 0.7510 \\ 0.2272 & 0.3142 \\ 0.4842 & 0.3740 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5190 & 0.3851 & 0.6518 & 0.1310 & 0.2305 \\ 0.5971 & 0.4729 & 0.5466 & 0.5970 & 0.5064 \\ 0.3805 & 0.3592 & 0.8039 & 0.2023 & 0.1848 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.1958 & 0.5790 & 0.8710 & 0.9427 & 0.0715 \\ 0.9716 & 0.8839 & 0.7459 & 0.6631 & 0.7721 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.6911 & 0.8366 \\ 0.9911 & 0.9238 \\ 0.1412 & 0.9555 \\ 0.7691 & 0.2172 \\ 0.4180 & 0.6685 \end{bmatrix}, \quad F = \begin{bmatrix} 0.7631 & 0.4555 \\ 0.0501 & 0.5027 \\ 0.6824 & 0.2716 \end{bmatrix}.$$

For the tracking problem, we considered the model

$$A_1 = \begin{bmatrix} 0.0119 & -0.7220 & 0.9915 \\ -0.4124 & 0.4140 & 0.2435 \\ -0.1547 & 0.2075 & 0.4138 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.6242 \\ 0.1253 \\ 0.1564 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.5635 & 0.6350 \\ 0.3515 & 0.9839 \\ 0.7593 & 0.2910 \end{bmatrix}$$

and used the initial feedback gain

$$K_0 = \begin{bmatrix} -1.8687 & -0.3008 & -1.5063 & 1.8896 & -0.7378 & 1.9863 & -0.5630 \\ 1.1583 & -1.9086 & -0.3427 & -1.2689 & -1.8124 & 1.0980 & -0.2509 \\ \\ -0.6677 & -0.5334 & -0.7351 & -0.8977 & 1.3538 & -1.6057 & 0.5989 \\ -0.1027 & -0.0875 & 0.7479 & 0.2787 & -1.6202 & 0.9261 & -0.2072 \end{bmatrix}.$$

For the model reference problem, we considered the model

$$A_1 \begin{bmatrix} -0.0313 & 0.2246 & 0.1190 \\ 0.0834 & -0.0586 & 0.2339 \\ 0.7635 & 0.0384 & -0.4704 \end{bmatrix}, B_1 = \begin{bmatrix} 0.6242 & 0.5635 \\ 0.1253 & 0.9838 \\ 0.1564 & 0.6350 \end{bmatrix}, C_1 = \begin{bmatrix} -0.5186 & 0.4181 \\ 0.4124 & 0.2971 \\ -0.0119 & 0.1547 \end{bmatrix}$$

and used the initial feedback gain matrix

$$K_0 = \begin{bmatrix} -0.5773 & -0.0756 & 0.3200 & 0.1270 & -0.3971 & -0.1547 & 0.9915 \\ 0.7430 & -0.7640 & 0.2484 & 0.9679 & -0.4181 & -0.7220 & 0.2435 \\ \\ 0.6301 & 0.0165 & -0.7493 & 0.2700 & 0.4124 & 0.4140 & 0.4138 \\ 0.7410 & -0.7173 & -0.6873 & 0.5186 & -0.4124 & 0.2075 & -0.9343 \end{bmatrix}.$$

Both initial gains were chosen at random. The variation of the four norms: the Hankel norm, the H_∞ norm, the FH norm, and the trace norm for the disturbance rejection problem was shown in Figure 2.9. Figure 2.13, and Figure 2.14, show analogous, results for the tracking

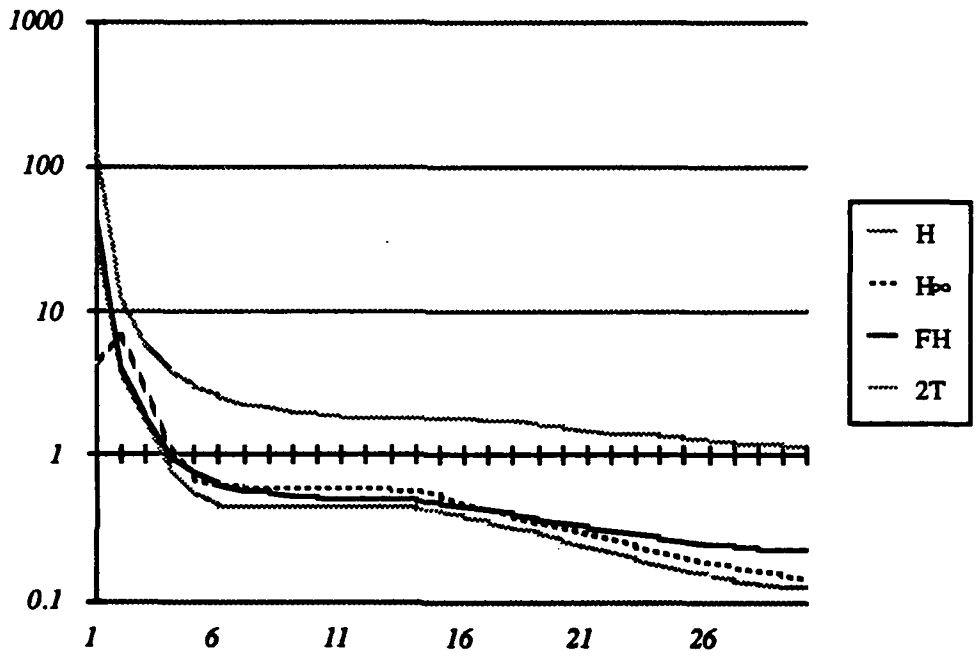

Figure 2.13: Log of various norms for the distribution of rejection problem.

and model reference problems, respectively. The value of K at iteration 40 for the disturbance rejection problem was given earlier. For the tracking problem at iteration 35 the gain was

$$K_{35} = \begin{bmatrix} -7.8413 & 5.3768 & -12.1889 & 1.4176 & 13.1417 & 0.3271 & 0.9358 \\ 8.1238 & -5.5467 & 7.7612 & -2.1177 & -9.7492 & -0.1666 & -0.9349 \\ 1.2912 & -0.9435 & 4.3515 & 0.9927 & -6.2684 & 0.6571 & -0.9840 \\ 1.2044 & -1.0983 & 0.9982 & 0.9183 & -2.1879 & 0.5003 & -0.2711 \end{bmatrix}.$$

and the value of K at iteration 40 for the model reference problem was

$$K_{40} = \begin{bmatrix} 2.4721 & -1.7562 & -15.1818 & 2.1323 & 16.6832 & -0.1850 & -1.8802 \\ -2.0549 & 1.6154 & 10.4600 & -2.7596 & -12.9371 & 0.3262 & 1.8608 \\ -1.8793 & 0.8583 & 8.8285 & 0.1290 & -11.7637 & 1.1118 & 1.4165 \\ 1.3085 & -0.0726 & -4.4312 & 0.1457 & 5.9159 & -0.6352 & -0.5588 \end{bmatrix}.$$

In this example, we can see that beyond 20 iterations, the various norms decrease exponentially. Figure 2.15 shows the convergence of the algorithm in each of the three problems as

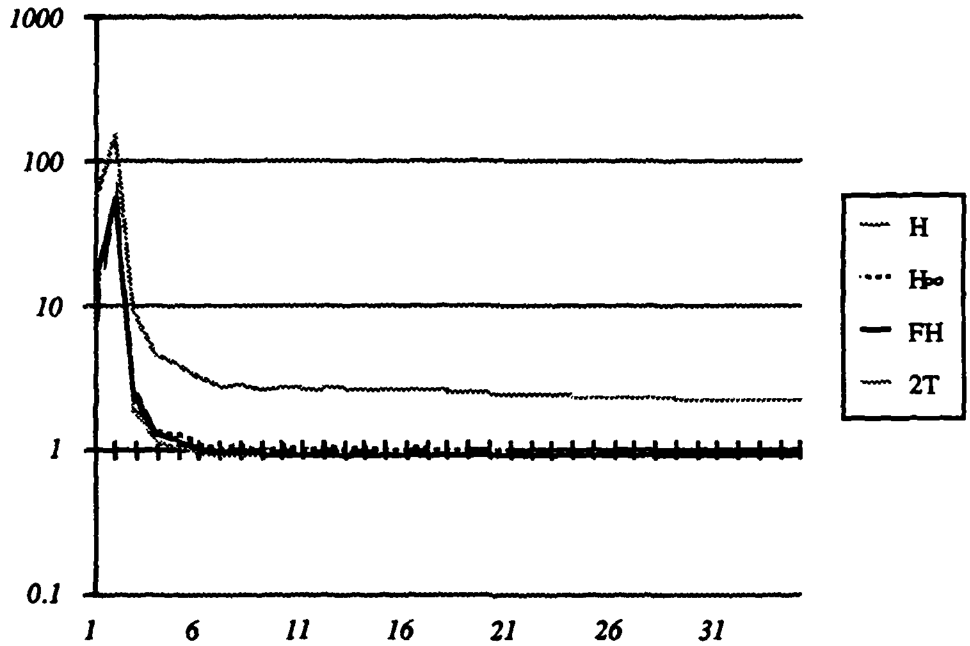

Figure 2.14: Log of various norms at each iteration for the model reference problem.

represented by the change in K_j. The FH norm and the H_∞ norm, in this example, are seen to be close not only at the optimum but also at each iteration. We can also see that in the example, the solutions are stabilizing, and also produce stable controllers.

Recall that in the disturbance-rejection example, (see Table 2.1) that the interval $\mathcal{I} =$

Table 2.1: Variations in ΔK_j for the three example problems.

Problem	$\mathcal{I}$-initial	H_∞-initial	$\mathcal{I}$-final	Optimized H_∞
Dis. Rej.	[6.00, 25.25]	9.44	[0.011, 0.040]	0.014
Tracking	[53.85, 154.07]	68.64	[0.90, 2.23]	0.94
Mod. Ref	[3.74, 12.28]	6.87	[0.012, 1.13]	0.014

[6.00, 25.25], bounds the H_∞ norm which was 9.44 at the point in which the first stabilizing gain was determined. After 30 iterations, the interval bounding the H_∞ norm was reduce to [.0105, .0399], and the actual values was .014. Corresponding results for the two other problems are summarized in Table 2.1. Thus, a significant reduction was made in the H_∞

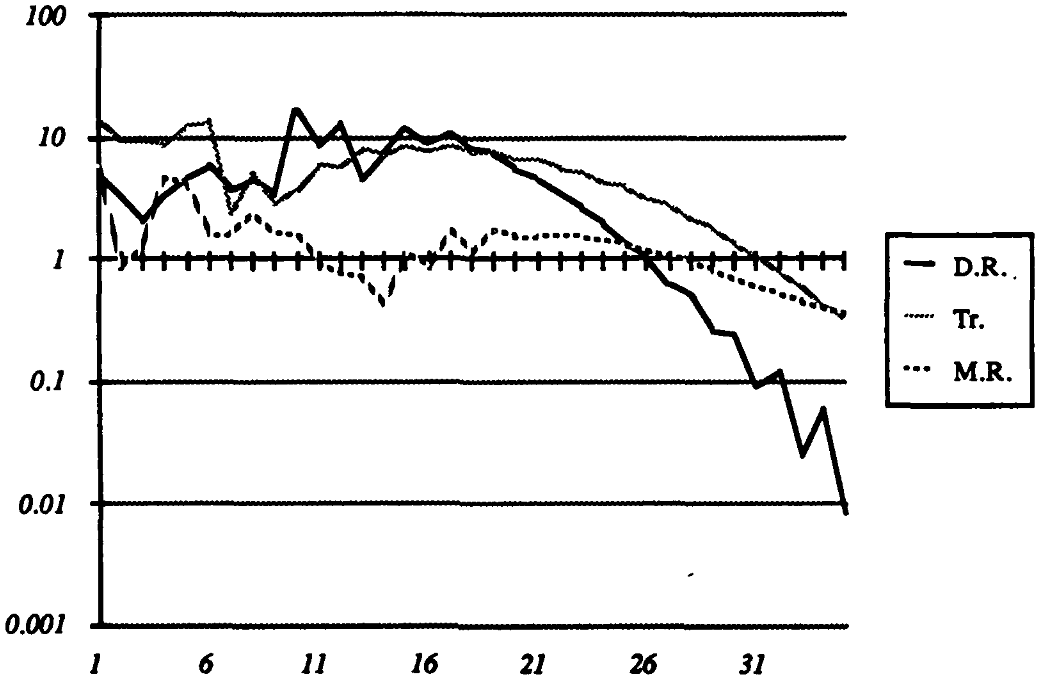

Figure 2.15: Log of various norms at each iteration for the tracking problem.

norm and also the bounding interval $\mathcal{I}$, in each of these examples.

The FH-norm approach to disturbance rejection provides computational ease and a near-optimal solution to the H_∞-norm minimization for controllers of bounded order. The ability to consider a broad class of problems makes this approach all the more attractive for control design. These features are amply illustrated on the fifth order example. The new Riccati equation based algorithm is computationally attractive since it takes advantage of developed computational tools for the Riccati equation and eliminates the search for an initial stabilizing solution.

3 LOW-ORDER CONTROLLER DESIGN USING PROJECTIVE CONTROLS

3.1 Time-Domain Properties of Projective Control

The projective controls approach offers a method of designing a low-order output feedback controller to retain a subset of eigenvalues and associated eigenvectors of a reference state-feedback system. The reference system is typically obtained using an LQ approach or an H_∞-norm approach. The obtained controller can be either static or dynamic, the order being determined in the design process so as to meet stated design objectives. Dynamic projective controllers are parameterized by a $p \times r$ matrix of free parameters, r being the dimension of the measured output vector, p being the order of the controller. When transient performance is the issue, an LQ approach is typically used to determine the reference system, and the retained eigenstructure is chosen to retain the dominant dynamics of the reference system. The design freedom available in the free parameters is then used to shape the residual dynamics. When disturbance rejection is the issue, an H_∞-norm approach is employed and the design freedom in the available free parameters is used to further improve disturbance rejection.

In this section time-domain properties of projective controls are reviewed emphasizing in particular a convenient parameterization of projective controllers. The remaining sections expand the projective controls methodology and provide design tools to achieve transient performance and disturbance rejection using low-order controllers. Section 3.2 concentrates on the problem of shaping the residual dynamics, Section 3.3 presents the frequency-domain properties of projective controls and their impact on the disturbance-rejection problem, Section 3.4 develops the FH norm approach to solve the disturbance-rejection problem using projective controllers, and Section 3.5 introduces a convenient similarity transformation which reduces the system representation to a linear in the free parameter form and extends the design to decentralized systems.

The projective controls method [29], [30] is a method for designing low-order controllers for higher-order systems based on retaining a subset of the poles and the associated eigenstructure of a reference system. The reference system is determined by a state-feedback controller which is chosen for its desirable properties. Many algorithms exist for designing state-feedback controllers; thus projective controls approach is suitable for use in combination with many types of synthesis methods. Moreover, once the state-feedback controller is determined, the projective controller is easily computed. In particular, this reference system can be written in the form

$$G_r(s) = G_{r1}(s)G_{r2}(s), \tag{3.1}$$

while the closed-loop projective controls system has been shown to reduce to

$$G_p(s) = G_{r1}(s)G_{p2}(s). \tag{3.2}$$

The $G_{r1}(s)$ is called the **retained** subsystem while $G_{p2}(s)$ is called the **residual** subsystem. The order of the retained subsystem is determined by the class of controller chosen. Three classes of controllers are considered here: static, proper and strictly proper controllers. For a static projective controller the residual dynamics are completely determined and stability and performance of the non-retained dynamics is not guaranteed. In the case of dynamic controllers, the projective controllers are parameterized by free parameters. These may be used to achieve stability and improve the performance of the residual subsystem. In the remainder of this section, we state the basic properties of projective controllers and develop controller parameterizations.

3.1.1 Static controllers

Suppose a state-feedback controller $u = K_o x$ is applied to the system (2.1) and yields the reference system

$$G_r(s) = (H + EK_0)(sI - F)^{-1}G, \tag{3.3}$$

where $F = A + BK_o$. The eigenstructure of the reference system is $FX = X\Lambda$ where Λ is a diagonal matrix of the eigenvalues of F, $\lambda(F)$ and X is a matrix of the associated eigenvectors. The reference system can be determined using any of the appropriate state-feedback design methodologies. One common design approach is LQ optimization. It has the desirable properties of producing controllers which are guaranteed to be stabilizing through the solution of the algebraic Riccati equation. In particular, the stabilizing controller which minimizes $\|G(s)\|_2$ is given by

$$u = K_2 x, \quad K_2 = -B^T M_2, \tag{3.4}$$

where $M_2 > 0$ is the solution of the algebraic Riccati equation

$$A^T M_2 + M_2 A - M_2 B B^T M_2 + H^T H = 0. \tag{3.5}$$

For details, see for example [31].

A stablizing controller which guarantees $\|G(s)\|_\infty \leq \gamma$ is given by

$$u = K_\infty x, \quad K_\infty = -B^T M_\infty \tag{3.6}$$

providing there exists $M_\infty > 0$ which satisfies the algebraic Riccati equation

$$A^T M_\infty + M_\infty A - M_\infty B B^T M_\infty + \frac{1}{\gamma^2} M_\infty G G^T M_\infty + H^T H = 0. \tag{3.7}$$

For details, see for example [1].

Consider now a static controller which retain the r reference eigenvalues Λ_r and associated eigenvectors X_r, where r is the number of measured outputs.

Theorem 3.1. *If Λ_r is observable from C, then the static output-feedback controller $C(s)$ retains $[\Lambda_r, X_r]$ if and only if*

$$C(s) = D_c \tag{3.8}$$

where

$$D_c = K_o N_o, \tag{3.9}$$

$$N_o \triangleq X_r (C X_r)^{-1}. \tag{3.10}$$

Proof. Let the feedback D_c retains $[\Lambda_r, X_r]$. We thus must have

$$A_c X_r = (A + B D_c C) X_r = X_r \Lambda_r. \tag{3.11}$$

Also

$$FX_r = (A + BK_o)X_r = X_r\Lambda_r. \tag{3.12}$$

Subtracting the two equations yields

$$BD_cCX_r = BK_oX_r. \tag{3.13}$$

Since Λ_r is observable from C, CX_r is invertible and (3.13) is satisfied by D_c given by (3.9), (3.10). Conversely, let D_c be given by (3.9), (3.10). Then

$$A_cX_r = AX_r + BK_0X_r = FX_r = X_r\Lambda r. \tag{3.14}$$

□

Theorem 3.2. *Given the control law (3.8)-(3.10), the eigenvalues of the closed-loop system are*

$$\lambda_c = \lambda_r \cup \lambda(A_r), \tag{3.15}$$

where

$$A_r \triangleq Y^T(I_n - N_oC)AY \tag{3.16}$$

and Y satisfies $CY = 0$ and $Y^TY = I_{n-r}$.

Proof. Let T be given by

$$T = [X_r \;\; Y], \quad T^{-1} = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} (CX_r)^{-1}C \\ Y^T(I_n - N_oC) \end{bmatrix}. \tag{3.17}$$

Note that U and V exist provided that CX_r is invertible which is guaranteed by the observability of Λ_r. Thus T is invertible since U and V exist.

$$T^{-1}A_cT = \begin{bmatrix} \Lambda_r & * \\ 0 & A_r \end{bmatrix} \tag{3.18}$$

with

$$A_r = Y^T(I_n - N_oC)AY \tag{3.19}$$

□

3.1.2 Proper Controllers

The following result identifies the class of p^{th}-order controllers that retain $r+p$ eigenvalues and the associated eigenvectors of the reference system.

Theorem 3.3. *The set of p^{th}-order proper controllers which retain $[\Lambda_r, X_r]$ and $[\Lambda_p, X_p]$ is given by*

$$C(s) = C_c(sI - A_c)^{-1}B_c + D_c \tag{3.20}$$

with $\{A_c, B_c, C_c, D_c\}$ parameterized by $P_o \in I\!R^{p\times r}$ as

$$\begin{array}{rcl} A_c & = & \Lambda_p + P_oCFB_o \\ B_c & = & P_oCF(N_o - B_oP_o) - \Lambda_pP_o \\ C_c & = & K_oB_o \\ D_c & = & K_o(N_o - B_oP_o) \end{array} \tag{3.21}$$

and $B_o \triangleq (I_n - N_oC)X_p$.

Proof. From

$$\tilde{A}_c\tilde{X}_r = \tilde{X}_r\tilde{\Lambda}_r \tag{3.22}$$

follows

$$\begin{bmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}\begin{bmatrix} X_p & X_r \\ W_p & W_r \end{bmatrix} = \begin{bmatrix} X_p & X_r \\ W_p & W_r \end{bmatrix}\begin{bmatrix} \Lambda_p & 0 \\ 0 & \Lambda_r \end{bmatrix}$$

or

$$\begin{array}{l} A_cW_p + B_cCX_p = W_p\Lambda_p \\ A_cW_r + B_cCX_r = W_r\Lambda_r \\ BC_cW_p + (A+BD_cC)X_p = X_p\Lambda_p = (A+BK_o)X_p \\ BC_cW_r + (A+BD_cC)X_r = X_r\Lambda_r = (A+BK_o)X_r \end{array} \tag{3.23}$$

or

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}\begin{bmatrix} W_p & W_r \\ CX_p & CX_r \end{bmatrix} = \begin{bmatrix} W_p\Lambda_p & W_r\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix};$$

thus,

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} W_p\Lambda_p & W_r\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix}\begin{bmatrix} W_p & W_r \\ CX_p & CX_r \end{bmatrix}^{-1}. \tag{3.24}$$

Define $L \triangleq W_p^{-1}W_r$; then (3.24) becomes

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} W_p\Lambda_p & W_pL\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix}\begin{bmatrix} W_p & W_pL \\ CX_p & CX_r \end{bmatrix}^{-1}$$

or

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} W_p & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \Lambda_p & L\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \begin{bmatrix} I_p & L \\ CX_p & CX_r \end{bmatrix}^{-1} \begin{bmatrix} W_p^{-1} & 0 \\ 0 & I_r \end{bmatrix} \tag{3.25}$$

Note that W_p represents a state-space transformation of the controller and thus W_p is arbitrary. It follows that

$$\begin{bmatrix} I_p & L \\ CX_p & CX_r \end{bmatrix}^{-1} = \begin{bmatrix} I_p + L\Delta^{-1}CX_p & -L\Delta^{-1} \\ -\Delta^{-1}CX_p & \Delta^{-1} \end{bmatrix}$$

with

$$\Delta \triangleq CX_r - CX_pL.$$

Defining $P_o \triangleq L(CX_r - CX_pL)^{-1}$, produces

$$\Delta^{-1} = (CX_r)^{-1}(I_r + CX_pP_o)$$

and

$$\begin{bmatrix} I_p & L \\ CX_p & CX_r \end{bmatrix}^{-1} = \begin{bmatrix} I_p + P_oCX_p & -P_o \\ -(CX_r)^{-1}CX_p(I_p + P_oCX_p) & (CX_r)^{-1}(I_r + CX_pP_o) \end{bmatrix}.$$

Now setting $W_p = (I_p + P_oCX_p)$, the identity in (3.24) becomes

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} (I_p + P_oCX_p)\Lambda_p & P_oCX_r\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \begin{bmatrix} I_p & -P_o \\ -(CX_r)^{-1}CX_p & (CX_r)^{-1}(I_r + CX_pP_o) \end{bmatrix}$$

or

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} \Lambda_p + P_oCF(X_p - N_oCX_p) & P_oCF(N_oCX_p - X_p)P_o + P_oCFN_o - \Lambda_pP_o \\ K_o(X_p - N_oCX_p) & K_o(N_oCX_p - X_p)P_o + K_oN_o \end{bmatrix},$$

which finally results in

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} \Lambda_p + P_oCFB_o & P_oCF(N_o - B_oP_o) - \Lambda_pP_o \\ K_oB_o & K_o(N_o - B_oP_o) \end{bmatrix}.$$

□

Theorem 3.4. *Given the control law (3.20)-(3.21), the eigenvalues of the closed-loop system are*

$$\tilde{\lambda}_c = \lambda_p \cup \lambda_r \cup \lambda(\tilde{A}_r) \tag{3.26}$$

where

$$\tilde{A}_r \triangleq A_r + B_o P_o A Y. \tag{3.27}$$

Proof: Consider

$$\tilde{A}_c = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$$

and define

$$\tilde{T} = \begin{bmatrix} X_p & X_r & Y \\ I_p + P_oCX_p & P_oCX_r & 0 \end{bmatrix},$$

which gives

$$\tilde{T}^{-1} = \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} = \begin{bmatrix} -P_oC & I_p \\ (CX_r)^{-1}C(I_n + X_pP_oC) & -(CX_r)^{-1}CX_p \\ Y^T + Y^T(B_oP_o - N_o)C & -Y^TB_o \end{bmatrix}.$$

It can now be verified that

$$\tilde{T}^{-1}\tilde{A}_c\tilde{T} = \begin{bmatrix} \Lambda_p & 0 & * \\ 0 & \Lambda_r & * \\ 0 & 0 & \tilde{A}_r \end{bmatrix}$$

with

$$\tilde{A}_r = Y^T(I_n + (B_oP_o - N_o)C)AY = A_r + Y^TB_oP_oAY.$$

□

3.1.3 Strictly proper controllers

Consider finally the p^{th}-order strictly proper controller which retain the p reference eigenvalues Λ_p and associated eigenvectors X_p.

Theorem 3.5. *The set of p^{th}-order strictly proper controllers which retain $[\Lambda_p, X_p]$ is given by*

$$C(s) = C_c(sI - A_c)^{-1}B_c \tag{3.28}$$

and parameterized by $P_o \in \mathbb{R}^{r\times p}$ where

$$\begin{aligned} A_c &= \Lambda_p - P_oCX_p \\ B_c &= P_o \\ C_c &= K_oX_p. \end{aligned} \tag{3.29}$$

Proof. From

$$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} X_p \\ W_p \end{bmatrix} = \begin{bmatrix} X_p \\ W_p \end{bmatrix} \Lambda_p \tag{3.30}$$

we have

$$A_cW_p + B_cCX_p = W_p\Lambda_p$$

or

$$\begin{aligned} A_c &= W_p\Lambda_pW_p^{-1} - B_cCX_pW_p^{-1} \\ &= W_p(\Lambda_p - W_p^{-1}B_cCX_p)W_p^{-1}. \end{aligned} \tag{3.31}$$

Defining $P_o \triangleq W_p^{-1}B_c$, relation (3.31) reduces to

$$A_c = W_p(\Lambda_p - P_oCX_p)W_p^{-1} \tag{3.32}$$

with

$$B_c = W_pP_o. \tag{3.33}$$

From (3.30) also follows

$$BC_cW_p + AX_p = X_p\Lambda_p$$

or

$$BC_cW_p + AX = (A + BK_o)X_p,$$

which is satisfied by

$$C_cW_p = K_oX_p$$

or

$$C_c = K_oX_pW_p^{-1}. \tag{3.34}$$

Note that W_p represents a state-space transformation of the controller and thus W_p is arbitrary. Choosing $W_p = I_p$, reduces (3.32)-(3.34) to (3.29).

Theorem 3.6. *Given the control law (3.28)-(3.29), the eigenvalues of the closed-loop system are*

$$\bar{\lambda}_c = \lambda_p \cup \lambda(\bar{A}_r) \tag{3.35}$$

where

$$\bar{A}_r \triangleq A - X_pP_oC. \tag{3.36}$$

Proof. Consider

$$\bar{A}_c = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$$

and introduce the transformation

$$\bar{T} = \begin{bmatrix} X_p & I_n \\ I_p & 0 \end{bmatrix}, \quad \bar{T}^{-1} = \begin{bmatrix} 0 & I_p \\ I_n & -X_p \end{bmatrix}.$$

Then

$$\bar{T}^{-1}\bar{A}_c\bar{T} = \begin{bmatrix} \Lambda_p & * \\ 0 & \bar{A}_r \end{bmatrix}$$

and so

$$\bar{A}_r = A - X_pP_oC.$$

□

3.1.4 Example

Consider the system defined by

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & -2 & 1 \\ -2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & -2 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \ \ 0 \ \ 0 \ \ 0], \\ Q &= \text{diag}\,\{100, 0, 100, 0\}, \quad R = 1, \end{aligned}$$

with

$$\text{Spec}\,\{A\} = \{-1.53 \pm j2.18, 0.53 \pm j0.92\}$$

and

$$\text{Spec}\,\{F\} = \{-3.57 \pm j4.09, -1.30, -0.56\}.$$

It can easily be determined in this problem that static projective controls will not stabilize the system. Thus, a first-order dynamic controller is sought. Since $r = 1$, $p = 1$, there is only one free design parameter. It effect on the residual dynamics can be observed by considering the root locus for the residual system (note that here $C = [I, \ \ 0]$, so $Y^T = [0 \ \ I_r]$)

$$\dot{x}_r = (A_r + B_0pA_{12})x_r$$

where

$$\begin{aligned} A_r &= \begin{bmatrix} -24.46 & 46.93 & -22.46 \\ 0.78 & 0.44 & 0.78 \\ 21.59 & -48.18 & 22.59 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -9.10 \\ 0.07 \\ 8.94 \end{bmatrix} \\ A_{12} &= [1 \quad -2 \quad 1]. \end{aligned}$$

Omitting the details, it is determined in this particular problem that the stabilizing values of p are in the interval [-2.575, -2.595]. The controller parameters are

$$\begin{array}{ll} H = -1.3 - 0.31p, & D = 0.85p + 0.31p^2 \\ N_d = 13.49, & K_d = -35.84 - 13.49p. \end{array}$$

Taking

$$p = -2.5875$$

the controller becomes

$$\begin{array}{ll} H = -0.50, & D = 0.17 \\ N_d = 13.49, & K_d = 0.49 \end{array}$$

and the spectrum of the closed loop system becomes

$$\text{Spec}\,\{A_{ce}\} = \{-0.56, -1.30, -0.20, -0.22 \pm j1.44\}.$$

Here, the first two eigenvalues have been retained from the reference dynamics, and the last three placed by solving the auxiliary pole-placement problem.

3.2 Shaping the Residual Dynamics

Consider presently that $C = [I_r \;\; 0]$ as in the previous example, and note that the residual dynamics (3.32) can be associated with an auxiliary static output-feedback control problem for a system of $(n-r)^{\text{th}}$-order with p input and q outputs, where $q =$ rank A_{12}. It is well known [32] that such an output pole-placement problem has a solution for almost all A_r, B_0 and A_{12}, and almost all desired spectra Λ_d if $n - r < p + q$, i.e., if

$$p > n - r - q.$$

This implies, in particular, that when A_{12} is maximum rank, $q = r$, the pole placement problem can be solved for almost all problems using an $(n-2r)^{\text{th}}$-order controller; this is of lower order than the Luenberger (minimum order) observer. We present a solution to the pole-placement problem in a novel way that utilizes the full available freedom, as opposed to earlier procedures where P is frequently (for ease of calculation) restricted to be of unity rank [33].

Let

$$T_1 = [U_1 \; U_2] \tag{3.37}$$

where $U_1 \in R^{(n-r)xr}$, $U_2 \in R^{(n-r)\times(n-2r)}$ satisfy

$$A_{12}U_1 = I \tag{3.38a}$$

$$A_{12}U_2 = 0. \tag{3.38b}$$

Then

$$A_1 = T_1^{-1}A_{re}T_1 = T_1^{-1}A_rT_1 + T_1^{-1}B_0P[I \; 0]. \tag{3.39}$$

Define

$$T_1^{-1}A_rT_1 = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \quad T_1^{-1}B_0 = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{3.40}$$

with $D_{11} \in R^{r\times r}$, $D_{21} \in R^{(n-2r)xr}$, $E_1 \in R^{r\times p}$. Then

$$A_1 = \begin{bmatrix} D_{11} + E_1P & D_{12} \\ D_{21} + E_2P & D_{22} \end{bmatrix}. \tag{3.41}$$

Now introduce the second transformation

$$T_2 = \begin{bmatrix} I_r & 0 \\ L & I \end{bmatrix}, \quad T_2^{-1} = \begin{bmatrix} I_r & 0 \\ -L & I \end{bmatrix}, L \in R^{(2n-r)xr}. \tag{3.42}$$

It can then be shown that

$$\begin{aligned} A_2 &= T_2^{-1}A_1T_2 = T_2^{-1}T_1^{-1}A_eT_1T_2 \equiv T^{-1}A_{re}T \\ &= \begin{bmatrix} D_{11} + D_{12}L + E_1P & D_{12} \\ -R(L) + (E_2 - LE_1) & D_{22} - LD_{12} \end{bmatrix} \end{aligned} \tag{3.43}$$

where

$$R(L) = LD_{11} - D_{22}L + LD_{12}L - D_{21}. \tag{3.44}$$

Now suppose $n - 2r > p$; this implies that a solution to the pole-placement problem almost always exists. It also allows a non-unity rank solution to the pole-placement problem to be determined. To this end, decompose $E_2 - LE_1$ as

$$E_2 - LE_1 = [M_1 \ M_2] \tag{3.45}$$

where $M_1 \in R^{(n-2r)\times(n-2r)}$ and $\det M_1 \neq 0$. Decompose P and E_1 as

$$P = [P_1 \ P_2], \quad E_1 = [E_1^a \ E_1^b] \tag{3.46}$$

with $P_1 \in R^{(n-2r)\times r}$, $P_2 \in R^{(p-n+2r)\times r}$, $E_1^a \in R^{r\times(n-2r)}$, and $E_1^b \in R^{r\times(p-n+2r)}$.

Theorem 3.7. *Let L place the pole of $D_{22} - LD_{12}$ at Λ_1 and let P_2 place the poles of $A_{c1} - B_{c1}P$ at Λ_2 where*

$$A_{c1} = D_{11} + D_{12}L + E_1^a M_1^{-1} R(L) \tag{3.47}$$

$$B_{c1} = E_1^a M_1^{-1} M_2 - E_1. \tag{3.48}$$

Then if

$$P_1 = M_1^{-1}(R(L) - M_2 P_2) \tag{3.49}$$

we have

$$\Lambda(A_{re}) = \Lambda_1 \cup \Lambda_2. \tag{3.50}$$

Proof. From 3.43, using (3.45) and (3.46) follows

$$A_2 = \begin{bmatrix} D_{11} + D_{12}L + E_1^a P_1 + E_1^b P_2 & D_{12} \\ -R(L) + M_1 P_1 + M_2 P_2 & D_{12} - LD_{12} \end{bmatrix}. \tag{3.51}$$

Choosing P_1 to satisfy (3.49) for given L and P_2 produces

$$A_2 = \begin{bmatrix} D_{11} + D_{12}L + E_1^a M_1^{-1} R(L) - (E_1^a M_1^{-1} M_2 - E_1^b) P_2 & D_{12} \\ 0 & D_{12} - LD_{12} \end{bmatrix}.$$

□

3.2.1 Example

To illustrate the above pole-placement procedure consider problem of placing the poles of

$$\begin{aligned}
\dot{x} &= \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -4 & -3 & -2 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hline 1 & 0 \end{array}\right] u + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] w \\
y &= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] x.
\end{aligned}$$

at $\Lambda_d = \{-3, -4, -5\}$. For this problem

$$\begin{aligned}
D_{11} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
D_{21} &= [-4 \ -3], \quad D_{22} = -2, \quad G_2 = [0], \\
B_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = [1 \ 0].
\end{aligned}$$

Follows

$$D_{22} - LD_{12} = -2 - [\ell_1 \ \ell_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2 - \ell_2.$$

Suppose the eigenvalue of $D_{22} - LD_{12}$ is selected to be $-4 \in \Lambda_d$. In view of above choose

$$\ell_2 = 2$$

and without loss of generality, let $\ell_1 = 0$. For this L we get

$$\begin{aligned}
R(L) &= LD_{11} - D_{22}L + LD_{12}L - D_{12} \\
&= [\ell_1 \ 2] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 2[\ell_1 \ 2] + [\ell_1 \ 2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\ell \ 2] - [-4 \ -3]
\end{aligned}$$

which produces

$$R(L) = [4 \ 3].$$

We now want to satisfy

$$R(L) = (B_2 - LB_1)P;$$

so,

$$\begin{aligned}
[4 \ 3] &= \left([1 \ 0] - [0 \ 2] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \\
&= [P_1 \ P_2] - 2[P_3 \ P_4],
\end{aligned}$$

and hence,

$$[P_1\ P_2] = [4\ 3] + 2[P_3\ P_4].$$

Since here $T_1 = I$, then $E = T_1^{-1}B = B$, and so $E_1 = B_1$, $E_2 = B_2$. Thus we solve the pole placement problem

$$\Lambda(D_{11} + D_{12}L + B_1K) = \Lambda_{d2} = \{-3, -5\}.$$

Since

$$\begin{aligned} D_{11} + D_{12}L + B_1P &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [0\ 2] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ P_3 & P_4 \end{bmatrix} \end{aligned}$$

this demands that

$$\begin{aligned} P_3 &= -15 \\ P_4 &= -8. \end{aligned}$$

Which in turn results in

$$\begin{aligned} P_1 &= -26 \\ P_2 &= -8. \end{aligned}$$

And so the gain matrices

$$P = -\begin{bmatrix} 26 & 17 \\ 15 & 8. \end{bmatrix}, \quad L = [0\ 2]$$

place the poles of A_{re} at $\{-3, -4, -5\}$.

3.3 Frequency Properties of Projective Controllers

In the previous sections, attention was focussed on improving the closed-loop performance by retaining properties of a state-feedback controlled system with a low-order projective controller; thus, time-domain properties of projective controls were exploited. Disturbance-rejection properties are typically judged according to frequency-domain measures such as the H_∞ norm. Expressions have therefore been developed which relate the frequency-domain properties of a system with a projective controller to the frequency-domain properties of a system with a state-feedback controller.

In treating frequency domain properties via transfer functions it is often useful to use the compact notation of the transfer function while retaining the state-space representation for computational purposes. We do so here and introduce the following notation for this purpose:

Definition 3.1. *Given the state-space representation of the LTI system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{3.52}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the input, $z \in \mathbb{R}^r$ is the output, the transfer function of the system shall be denoted by the packed representation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq C(sI - A)^{-1}B + D. \tag{3.53}$$

Thus, through the use of the expression (3.53), the transfer function of the system is represented in terms of a state-space representation using a compact notation.

Let

$$G_1(s) \triangleq \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad G_2(s) \triangleq \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]. \tag{3.54}$$

If the two systems $G_1(s)$ and $G_2(s)$ are cascaded together as in Figure 3.1, the resulting

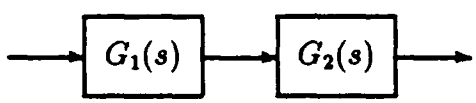

Figure 3.1: Cascade connection.

system can be represented as

$$G_2(s)G_1(s) = \left[\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right]. \tag{3.55}$$

If the two systems are connected in parallel as in Figure 3.2, the resulting system can be represented as

$$G_1(s) + G_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]. \tag{3.56}$$

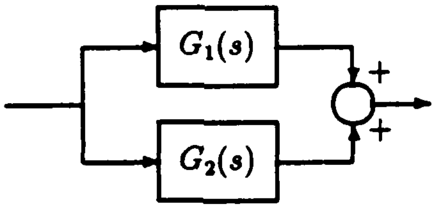

Figure 3.2: Parallel connection.

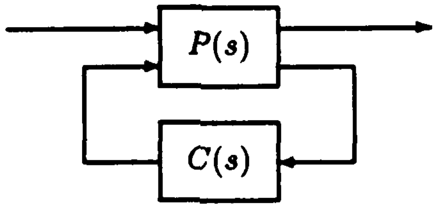

Figure 3.3: Feedback connection.

Note that representations (3.55) and (3.56) are not necessarily minimal. Consider finally the feedback configuration of Figure 3.3. Let the plant have the state-space representation

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ z &= Hx + Eu \\ y &= Cx + Dw \end{aligned} \tag{3.57}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the disturbance input, $u \in \mathbb{R}^r$ is the controlled input, $z \in \mathbb{R}^s$ is the controlled output, $y \in \mathbb{R}^m$ is the measured output. The transfer function $P(s)$ is characterized by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}. \tag{3.58}$$

Then

$$P(s) \triangleq \left[\begin{array}{c|cc} A & G & B \\ \hline H & 0 & E \\ C & D & 0 \end{array}\right]. \tag{3.59}$$

If the system is controlled by

$$C(s) \triangleq \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{3.60}$$

so that

$$u(s) = C(s)y(s), \tag{3.61}$$

then the closed loop feedback system represented on Figure 3.2 is given by

$$G(s) = \left[\begin{array}{cc|c} A + BD_cC & BC_c & G + BD_cD \\ B_cC & A_c & B_cD \\ \hline H + ED_cC & EC_c & ED_cD \end{array}\right]. \tag{3.62}$$

We can now state the following results for transfer function properties of projective controllers.

Theorem 3.8. *Define the error between the static projective system and the reference system as*

$$E(s) \triangleq G_r(s) - G_p(s). \tag{3.63}$$

Then

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} A_r & G_r \\ \hline K_oY & 0 \end{array}\right] = E_1(s) \cdot E_2(s) \tag{3.64}$$

where $G_r \triangleq Y^T(I_n - N_oC)G$.

Proof. Recall here that $\tilde{A}_r$ is given by (3.36). By definition, $E(s)$ is given by

$$E(s) = \left[\begin{array}{cc|c} F & 0 & G \\ 0 & A_c & -G \\ \hline H + EK_o & H + ED_cC & 0 \end{array}\right]. \tag{3.65}$$

Applying the state space transformation

$$T_o \triangleq \begin{bmatrix} I_n & -X_r & -Y \\ 0 & X_r & Y \end{bmatrix}, \quad T_o^{-1} = \begin{bmatrix} I_n & I_n \\ 0 & U \\ 0 & V \end{bmatrix} \tag{3.66}$$

produces

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & -BK_oY & 0 \\ 0 & \Lambda_r & UAY & -UG \\ 0 & 0 & \tilde{A}_r & -VG \\ \hline H + EK_o & 0 & -EK_oY & 0 \end{array}\right]. \tag{3.67}$$

Removing the unobservable states yields

$$E(s) = \left[\begin{array}{cc|c} F & -BK_oY & 0 \\ 0 & A_r & -VG \\ \hline H + EK_o & -EK_oY & 0 \end{array}\right] = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} A_r & VG \\ \hline K_oY & 0 \end{array}\right]. \tag{3.68}$$

□

Theorem 3.9. *Define the error between the proper projective system and the reference systems as* $E(s) \triangleq G_r(s) - G_p(s)$. *Then*

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \tilde{A}_r & G_r + Y^T B_o P_o C G \\ \hline K_o Y & 0 \end{array}\right] = E_1(s) \cdot E_2(s; P_o). \tag{3.69}$$

Proof. Recall that $\tilde{A}_r$ is given in (3.27). By definition now

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & 0 & -G \\ 0 & A + BD_c C & BC_c & G \\ 0 & B_c C & A_c & 0 \\ \hline H + EK_o & H + ED_c C & EC_c & 0 \end{array}\right]. \tag{3.70}$$

Applying the state space transformation

$$\tilde{T}_o \triangleq \begin{bmatrix} I_n & -X_p & -X_r & -Y \\ 0 & X_p & X_r & Y \\ 0 & I_p + P_o C X_p & P_o C X_r & 0 \end{bmatrix} \tag{3.71}$$

$$\tilde{T}_o^{-1} = \begin{bmatrix} I_n & I_n & 0 \\ 0 & -P_o C & I_p \\ 0 & (CX_r)^{-1} C (I_n + X_p P_o C) & -(CX_r)^{-1} C X_p \\ 0 & Y^T + Y^T (B_o P_o - N_o) C & -Y^T B_o \end{bmatrix} \tag{3.72}$$

yields

$$E(s) = \left[\begin{array}{cccc|c} F & 0 & 0 & -BK_o Y & 0 \\ 0 & \Lambda_p & 0 & * & * \\ 0 & 0 & \Lambda_r & * & * \\ 0 & 0 & 0 & \tilde{A}_r & G_r + Y^T B_o P_o C G \\ \hline H + EK_o & 0 & 0 & -EK_o Y & 0 \end{array}\right]. \tag{3.73}$$

Removing the unobservable states yields

$$E(s) = \left[\begin{array}{cc|c} F & -BK_o Y & 0 \\ 0 & \tilde{A}_r & G_r + Y^T B_o P_o G \\ \hline H + EK_o & -EK_o Y & 0 \end{array}\right], \tag{3.74}$$

which is equivalent to

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \tilde{A}_r & G_r + Y^T B_o P_o C G \\ \hline K_o Y & 0 \end{array}\right], \tag{3.75}$$

Theorem 3.10. *Define the error between the strictly proper projective system and the reference system as* $E(s) \triangleq G_r(s) - G_p(s)$. *Then*

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \bar{A}_r & G \\ \hline K_o & 0 \end{array}\right]. \tag{3.76}$$

Proof. By definition, $E(s)$ is now given by

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & 0 & -G \\ 0 & A & BC_c & G \\ 0 & B_cC & A_c & 0 \\ \hline H + EK_o & H & EC_c & 0 \end{array}\right]. \tag{3.77}$$

Applying the state space transformation

$$\bar{T}_o \triangleq \begin{bmatrix} I_n & -X_p & -I_n \\ 0 & X_p & I_n \\ 0 & I_p & 0 \end{bmatrix}, \quad \bar{T}_o^{-1} = \begin{bmatrix} I_n & I_n & 0 \\ 0 & 0 & I_p \\ 0 & I_n & -X_p \end{bmatrix} \tag{3.78}$$

yields

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & -BK_o & 0 \\ 0 & \Lambda_p & P_0C & 0 \\ 0 & 0 & \bar{A}_r & G \\ \hline H + EK_o & 0 & -EK_o & 0 \end{array}\right]. \tag{3.79}$$

Removing the unobservable states yields

$$E(s) = \left[\begin{array}{cc|c} F & -BK_o & 0 \\ 0 & \bar{A}_r & G \\ \hline H + EK_o & -EK_o & 0 \end{array}\right] = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \bar{A}_r & G \\ \hline K_o & 0 \end{array}\right].$$

3.4 FH-Norm Optimization of Projective Systems

Previous development has shown that, when dynamic projective controls are used, the error transfer function in all cases reduces to

$$E(s) = E_1(s) \cdot E_2(s),$$

where $E_1(s)$ is independent of the free controller parameter while $E_2(s)$ depends on P_0. Because $\|E(s)\| \leq \|E_1(s)\| \cdot \|E_2(s)\|$ and $\|E_1(s)\|$ is constant, it is a natural idea is to choose P_0 to reduce $\|E_2(s)\|$. This, however, is not necessary and one may attempt to reduce $\|E(s)\|$ which represents a frequency weighted optimization problem with respect to P_0. In either case, an auxiliary minimization problem is solved to determine the free parameters of the dynamic projective controllers in the disturbance-rejection problem. The auxiliary minimization problem is to find $\bar{P}_0$ to solve

$$\bar{P}_0 = \arg \min_{P_o} \|T(s; P_o)\|, \tag{3.80}$$

where $T(s;P_o)$ is some appropriate transfer function which is dependent upon P_o, and $\|\cdot\|$ is an appropriate norm. The transfer functions that we will consider here are $E_2(s)$ given in (3.64) or (3.69), and the norm will be the FH norm as the computationally feasible alternative. The alternative approach where $T(s) = E(s)$ is also of interest and has the advantage that it takes into account the frequency weighting implied by $E_1(s)$ in (3.64) or (3.69). Finally, we may choose

$$T(s) = G(s) \tag{3.81}$$

in which case the intent is not to reduce the error, but instead directly reduces $\|G(s)\|$ subject to constraint on controller structure. In this section we choose to select the free parameters of the system P_o to satisfy

$$\bar{P}_0 = \arg \min_{P_o} \|E_2(s;P_o)\|_{FH}, \tag{3.82}$$

while the alternative approach of reducing $\|G(s)\|_{FH}$ is considered in the next section.

For the case of strictly proper controllers $E_2(s)$ is given by

$$E_2(s) = \left[\begin{array}{c|c} \bar{A}_r & G \\ \hline K_o & 0 \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_r & B_r \\ \hline C_r & 0 \end{array}\right], \tag{3.83}$$

where $\bar{A}_r$ is given by (3.36). The problem then is to minimize over P_o

$$J(P_o) = \text{ Tr } P_r Q_r \tag{3.84}$$

subject to the two Lyapunov equations

$$A_r P_r + P_r A_r^T + B_r B_r^T = 0 \tag{3.85}$$

$$A_r^T Q_r + Q_r A_r + C_r^T C_r = 0$$

which define the controllability and observability grammians of (3.83). The necessary conditions for a minimum then consists of (3.85) and

$$\begin{array}{lcl} A_r L_r + L_r A_r^T + P_r & = & 0 \\ A_r^T M_r + M_r A_r + Q_r & = & 0 \\ X_p^T(Q_r L_r + M_r P_r)C^T & = & 0 \end{array} \tag{3.86}$$

where M_r and L_r are Lagrange multipliers for the constraints (3.85).

The gradient of J with respect to P_o can be computed for arbitrary P_o as

$$\frac{dJ}{dP_o} = -2X_p^T(Q_rL_r + M_rP_r)C^T \tag{3.87}$$

where P_r, Q_r, L_r and M_r solve the Lyapunov equations (3.86).

Thus, a feasible directions algorithm can be implemented by iteratively solving

$$P_o^{i+1} = P_o^i - \epsilon\left(\frac{dJ}{dP_o}\right) \tag{3.88}$$

for the optimal P_o.

For proper controllers, we will assume for simplicity that $C = [I \quad 0]$, so $Y = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and the expression for $E_2(s)$ reduces to

$$E_2(s) = K_2(sI - \tilde{A}_r)^{-1}(VG + \tilde{B}_0P_0G_1) \tag{3.89}$$

where $\tilde{A}_r$ is given by (3.27), $K_2 = K_0Y$, V is defined in (3.17), $\tilde{B}_0 = Y^TB_0$ and $G_1 = CG$. The necessary conditions then take the form

$$\begin{aligned} &\tilde{A}_r^TQ_r + Q_r\tilde{A}_r + K_2^TK_2 = 0 \\ &\tilde{A}_rP_r + P_r\tilde{A}_r + (VG + \tilde{B}_0P_0G_1)(VG + \tilde{B}_0P_0G_1)^T \\ &\tilde{A}_rLr + L_r\tilde{A}_r + P_r = 0 \\ &\tilde{A}_r^TM_r + M_r\tilde{A}_r + Q_2 = 0 \end{aligned} \tag{3.90}$$

and

$$\frac{\partial J}{\partial P_0} = 2\tilde{B}_0^T[Q_rM_r + L_rP_r]A_{12}^T + 2\tilde{B}_0^TM_r(VG + \tilde{B}_0P_0G_1)G_1^T = 0. \tag{3.91}$$

The structure of (3.91) allows the use of the steepest descent method, with P_0 adjusted via (3.88), as well as the use of the Riccati equation based algorithm when $(G_1G_1^T)^{-1}$ exists. The next iterate for P_0 is then given by

$$P_0^{i+1} = -(\tilde{B}_0^TM_r^i\tilde{B}_0)^{-1}[\tilde{B}_0^T(Q_r^iM_r^i + L_r^iP_r^i)A_{12}^T + \tilde{B}_0^TM_r^iVGG_1^T](G_1G_1^T)^{-1}. \tag{3.92}$$

3.5 Disturbance Rejection using the FH Norm and Projective Controllers

3.5.1 Problem formulation

In some cases, it may be important to consider directly disturbance rejection with respect to $G(s)$, where $G(s)$ is the closed-loop transfer function of the system using proper or strictly proper projective controllers. It is, however, observed from the parameterization of the projective controllers that the closed-loop system is a non-linear (quadratic) function of the free parameter matrix P_0. Thus to simplify the computational issues, and reduce in the design phase the system representation to simplest form, we seek in this section a reduction to LIFP (linear-in-the-free-parameters) representation of the closed loop system. We will do so in the decentralized setting where a number of decentralized low-order projective controllers is used to retain by joint action a selected invariant subspace. In the decentralized case, however, even the residual dynamics exhibits a nonlinear dependence on the free design parameters $P_1, \ldots, P_q$, where q is the number of decentralized control channels and P_i is the free parameter matrix parameterizing the i-th controller. Thus, in the decentralized case, it is even more significant to reduce the system to the LIFP representation. We therefore develop the LIFP representation here for the decentralized control problem, which reduces to the centralized problem when $q = 1$.

Consider the decentralized system of Figure 3.4. The *state space description* of this system can be written as follows:

$$\begin{aligned} \dot{x} &= Ax + B_1 u_2 + B_2 u_2 + Gw \\ y_1 &= C_1 x \\ y_2 &= C_2 x \\ y_c &= Hx. \end{aligned} \tag{3.93}$$

where $x \in I\!R^n$, $u_1, u_2 \in I\!R^M$, $y_1, y_2 \epsilon I\!R^r$, $y_c \epsilon I\!R^s$, and $w \epsilon I\!R^q$. Let the dynamic controllers have the structure

$$\begin{aligned} \dot{\xi}_i &= H_i \xi_i + D_i y_i \\ u_i &= -N_{di} \xi_i - K_{di} y_i, \quad i = 1, 2. \end{aligned} \tag{3.94}$$

where $\xi_i \in I\!R^P$, $i = 1, 2$, and define the extended system as

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_{1e} u_1 + B_{2e} u_2 + G_e w \\ y_{1e} &= C_{1e} x_e \\ y_{2e} &= C_{2e} x_e \\ y_{ce} &= H_e x_e \end{aligned} \tag{3.95}$$

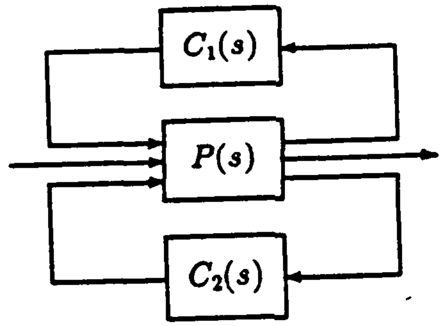

Figure 3.4: Decentralized system.

where

$$
\begin{aligned}
A_c &= \begin{bmatrix} A & 0 & 0 \\ D_1C_1 & H_1 & 0 \\ D_2C_2 & 0 & H_2 \end{bmatrix}, B_{1e} = \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, B_{2e} = \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix} \\
C_{1e} &= \begin{bmatrix} C_1 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, C_{2e} = \begin{bmatrix} C_2 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\
G_e &= \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix}, H_e = [H \quad 0 \quad 0].
\end{aligned}
\tag{3.96}
$$

The goal is to determine $\{H_1, D_1, N_{d1}, K_{d1}\}$ and $\{H_2, D_2, N_{d2}, K_{d2}\}$ to achieve certain performance and disturbance rejection goals.

It has been shown [34] that if

$$u = -K_0 x \tag{3.97}$$

is a stable feedback control producing the closed-loop system

$$
\begin{aligned}
\dot{x} &= Fx + Gw \\
y_c &= Hx
\end{aligned}
\tag{3.98}
$$

with F having the Jordan decomposition

$$F[x_{r+p}\ x_c] = [x_{r+p}\ x_c]\begin{bmatrix} \Lambda_{r+p} & 0 \\ 0 & \Lambda_c \end{bmatrix}, \quad X_{r+p} \in E^{n(r+p)}, \tag{3.99}$$

then there exist dynamic controllers of the form (3.94) such that the resulting closed-loop system retains all eigenvalues in Λ_{r+p} together with the associated invariant spaces. In fact, the entire family of such controllers has been parameterized [34] as follows:

$$\begin{aligned} H_i &= \tilde{W}_{pi}\tilde{H}_i\tilde{W}_{pi}^{-1}, \quad \tilde{H}_i = \Lambda_{pi} + P_i F_{12}^i \tilde{B}_{0i}, \quad q_i = 1,2 \\ D_i &= \tilde{W}_{pi}\tilde{D}_i, \quad \tilde{D}_i = P_i F_r^i - \tilde{H}_i P_i, \quad i = 1,2 \\ K_{di} &= K_{si} - N_{di}P_i, \quad i = 1,2 \\ N_{di} &= \tilde{N}_{di}\tilde{W}_{pi}^{-1}, \quad \tilde{N}_{di} = K_2^i \tilde{B}_0^i, \quad i = 1,2. \end{aligned} \tag{3.100}$$

Here P_i, $i = 1,2$ are free parameter matrices of dimension $p \times r$, the presence of $\tilde{W}_{pi}$ implies the invariance to similarity transformations, while Λ_p, Λ_r are partitions of the Jordan form Λ_{r+p} and F_{12}^i, $\tilde{B}_0^i$, F_r^i, K_{si}, N_{di} and $K_2^i, i = 1,2$ are known quantities determined directly by the reference solution F and its eigenvectors, assuming for each i that the system has transformed into the representation where $C_i = [I_r\ 0]$. For details see [34].

If transient performance is of primary concern, then the reference state-feedback solution can be determined by solving an LQ optimization problem, and projective controls will then retain in the closed-loop system the dominant poles of the reference solution that define acceptable transient response. The free parameters are then determined by solving an auxiliary problem to shape the residual dynamics.

If disturbance rejection and transient response are of concern, then the reference solution can be determined to minimize the H_∞ norm. The H_∞-optimal state-feedback control is given again by (3.6) where $M > 0$ is the solution of the ARE (3.7) with $B = [B_1\ \ B_2]$, and γ is the minimal value for which $M > 0$ solving (3.6) exists. The use of (3.6) guarantees here

$$\|G_c(s)\|_\infty = \|H(sI - F)^{-1}G\|_\infty \leq \gamma \tag{3.101}$$

with both control channels using state-feedback controls. Projective controls will now fix the dominant poles and associated eigenvectors of the system at locations determined by the reference solution, while the free parameters P_1, P_2 are to be used to shape the residual dynamics to achieve disturbance rejection. To simplify the disturbance rejection problem in applying the FH-norm minimization approach, the transformation developed in [34] that reduces the closed-loop to a linear-in-the-free-parameters (LIFP) form will be applied.

3.5.2 Transformation to the LIFP form

Assuming, without loss of generality, that $\tilde{W}_{pi} = I$, $\tilde{W}_{p2} = I$, the closed-loop system becomes

$$\dot{x}_e = A_{ce}x_e + G_e w \tag{3.102}$$

$$y_{ce} = H_e x_e, \tag{3.103}$$

with

$$A_{ce} = \begin{bmatrix} A_d & -B_1N_{d1} & -B_2N_{0d2} \\ (P_1F_r^1 - H_1P_1)C_1 & \Lambda_{pl} + P_1F_{12}\tilde{B}_0^1 & 0 \\ (P_2F_r^2 - H_2P_2)C_2 & 0 & \Lambda_{p2} + P_2F_{12}^2\tilde{B}_0^2 \end{bmatrix} \tag{3.104}$$

where

$$A_d = A_c + B_1N_{d1}P_1 + B_2N_{d2}P_2, \quad A_c = A - B_1K_{s1}C_1 - B_2K_{s2}C_2. \tag{3.105}$$

Now apply the transformation $\tilde{x}_3 = \tilde{T}\tilde{x}_e$ where

$$\tilde{T} = \begin{bmatrix} I_n & 0 & 0 \\ P_1C_1 & I_p & 0 \\ P_2C_2 & 0 & I_{p2} \end{bmatrix}, \quad \tilde{T}^{-1} = \begin{bmatrix} I_n & 0 & 0 \\ -P_1C_1 & I_p & 0 \\ -P_2C_2 & 0 & I_{p2} \end{bmatrix}. \tag{3.106}$$

The system (3.103) becomes

$$\begin{aligned} \dot{\tilde{x}}_3 &= \tilde{A}_{ce}\tilde{x}_3 + \tilde{G}_e w \\ \tilde{y}_{ce} &= \tilde{H}_e\tilde{x}_e, \end{aligned} \tag{3.107}$$

where

$$\tilde{A}_{ce} = \tilde{T}^{-1}A_{ce}\tilde{T}, \quad \tilde{G}_e = \tilde{T}^{-1}G_e, \quad \tilde{H}_e = H_e\tilde{T}. \tag{3.108}$$

The expression for $\tilde{A}_{ce}$ can thus be derived to be

$$\begin{aligned} \tilde{A}_{ce} &= \begin{bmatrix} A_c & -B_1N_{d1} & B_2N_{d2} \\ P_1E_1 & \Lambda_p + P_1G_{11} & P_1G_{12} \\ P_2E_2 & P_2G_{21} & \Lambda_{p2} + P_2G_{22} \end{bmatrix} \\ &= \tilde{A}_e + \tilde{B}_{1e}P_2\tilde{C}_{1e} + \tilde{B}_{2e}P_2\tilde{C}_{2e} \end{aligned} \tag{3.109}$$

where

$$\tilde{A}_e = \begin{bmatrix} A_c & -B_1N_{d1} & -B_2N_{d2} \\ 0 & \Lambda_p & 0 \\ 0 & 0 & \Lambda_{p2} \end{bmatrix} \quad \tilde{B}_{1e} = \begin{bmatrix} 0 \\ I_{p1} \\ 0 \end{bmatrix} \quad \tilde{B}_{2e} = \begin{bmatrix} 0 \\ 0 \\ I_{p2} \end{bmatrix}, \tag{3.110}$$

$$C_{1e} = [E_1 \ \ G_{11} \ \ G_{12}], \quad C_{2e} = [E_2 \ \ G_{21} \ \ G_{22}]$$

with

$$\begin{aligned} E_1 &= F_r^1C_1 - C_1A_c, \quad E_2 = F_r^2C_2 - C_2A_c \\ G_{11} &= F_{12}^2B_0^1 + C_1B_1N_{d1}, \quad G_{12} = C_1B_2N_{d2} \\ G_{21} &= C_2B_1N_{d1}, \quad G_{22} = F_{12}^2B_0^2 + C_2B_2N_{d2} \end{aligned} \tag{3.111}$$

while

$$\begin{aligned} \tilde{G}_e &= \tilde{T}^{-1}G_e = \begin{bmatrix} I_n & 0 & 0 \\ -P_1C_1 & I_p & 0 \\ -P_2C_2 & 0 & I_{p2} \end{bmatrix} \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} G \\ -P_1C_1G \\ -P_2C_2G \end{bmatrix} \\ &= G_3 - \tilde{B}_{1e}P_1C_1G - \tilde{B}_{2e}P_2C_2G \end{aligned} \tag{3.112}$$

and

$$\tilde{H}_e = H_e\tilde{T} = H_e. \tag{3.113}$$

Thus, when the similarity transformation is applied to the system, the expression derived for $\tilde{A}_{ce}$, $\tilde{G}_e$, and $\tilde{H}_e$ all exhibit a *linear* dependence on the free parameter matrices P_1 and P_2. ($\tilde{H}_e$ is in fact independent of the free parameter matrices.) This linear dependence can now be utilized to determine suitable P_1 and P_2 (and thus the dynamic controllers) to achieve disturbance rejection by minimizing the FH norm.

3.5.3 FH-norm minimization

This minimization now reduces to the minimization of $J_e =$ Trace P_eQ_e subject to the following constraints:

$$\begin{aligned} \tilde{A}_{ce}^TQ_e + Q_e\tilde{A}_{ce} + \tilde{H}_e^T\tilde{H}_e &= 0 \\ \tilde{A}_{ce}P_e + P_e\tilde{A}_{ce}^T + \tilde{G}_e\tilde{G}_e^T &= 0. \end{aligned} \tag{3.114}$$

By defining the Lagrange multipliers, $L_e = L_e^T$, $M_e = M_e^T$, the problem can again be reduced to an unconstrained minimization and since $\tilde{A}_{ce}$, $\tilde{G}_e$, and $\tilde{H}_e$ are all linear functions of the free parameters; only minor changes are introduced in the usual necessary conditions. These can now be written as follows:

$$\begin{aligned}
\partial J/\partial P_e &= \tilde{A}_{ce}^T M_e + M_e \tilde{A}_{ce} + Q_e = 0 \\
\partial J/\partial Q_e &= \tilde{A}_{ce} L_e + L_e \tilde{A}_{ce}^T + P_e = 0 \\
\partial J/\partial L_e &= \tilde{A}_{ce}^T Q_e + Q_e \tilde{A}_{ce} + \tilde{H}_e^T \tilde{H}_e = 0 \\
\partial J/\partial M_e &= \tilde{A}_{ce} P_e + P_e \tilde{A}_{ce}^T + \tilde{G}_e \tilde{G}_e^T = 0 \\
\partial J/\partial P_1 &= 2\tilde{B}_{1e}^T (M_e P_e + Q_e L_e) \tilde{C}_{1e}^T - 2\tilde{B}_{1e}^T M_e G_e G^T C_1^T \\
\partial J/\partial P_2 &= 2\tilde{B}_{2e}^T (M_e P_e + Q_e L_e) \tilde{C}_{2e}^T - 2\tilde{B}_{2e}^T M_e G_e G^T C_2^T .
\end{aligned} \tag{3.115}$$

The feasible direction algorithm can now be applied. Initially, the free parameter matrices, P_1 and P_2, are set to zero, (although arbitrary values can be used). If the resulting closed-loop system matrix, $\tilde{A}_{ce}$ given by (3.104) has unstable eigenvalues, J_e is not defined. In this case, an embedding parameter, p, is chosen such that

$$p > \mathrm{Max}_i(Re\lambda_i(A)), \tag{3.116}$$

and $\tilde{A}_{ce}$ modified to $\tilde{A}_{ce} - pI$. If the resulting $\tilde{A}_{ce}$ is stable, the embedding parameter is then zero. This leads to a modified extension J_{ea} with an expanded region of definition encompassing the initial P_1, P_2.

In the feasible direction algorithm, the first four equations in (3.115) are solved for M_e, L_e, Q_e and P_e. These are then used in the last two equations to calculate a gradient direction for the next iterate of P_1 and P_2, with

$$\begin{aligned}
P_1^{i+1} &= P_1^i - s\partial J/\partial P_1 \\
P_2^{i+1} &= P_2 - s\partial J/\partial P_2,
\end{aligned} \tag{3.117}$$

where s is the step size solving the one dimensional minimization problem

$$s = \arg\min_{h>0} J_{ea}(P_1 - h\frac{\partial J}{\partial P_1}, P_2 - h\frac{\partial J}{\partial P_2}) \tag{3.118}$$

thus guaranteeing convergence to a local minimum. Using the embedding parameter method, the maximal eigenvalue of $\tilde{A}_{ce}$ can be successively moved towards the imaginary axis. The parameter s can then be decreased and ultimately when the system is stabilized s can be set to zero resulting in the minimization of the original criterion J_e. However, it may not be possible to move all unstable eigenvalues into the left-half plane without simultaneously forcing previously stable eigenvalues into the right-half plane. If this situation occurs, the order of the dynamic controller must be increased to provide additional design freedom needed to stablize the system. Expanding controllers if necessary is simple in view of the way controllers are parameterized (see [34]) and expanded free parameter matrices can utilize the latest iterates of P_1 and P_2 to simply continue the combined stabilization and optimization process. This resulting algorithm applicable to an arbitrary number of controllers can be summarized as follows:

1) Initialize $P_i^1 =, i = 1, \ldots, k$.

2) Evaluate the resulting closed-loop system matrix $\tilde{A}_{ce}$, based on the current iterative value of P_i, $i = 1, \ldots, k$. If $\tilde{A}_{ce}$ is stable, proceed to step 5.

3) Choose an embedding parameter, p, such that $p > \text{Max}_i(Re\Lambda_i(A))$.

4) Solve the associated minimization problem recursively until p can be set equal to zero. (If $\tilde{A}_{ce}$ cannot be stabilized, increase the order of the dynamic controllers to be used and start the algorithm over.)

5) Solve for P_e, Q_e, L_e and M_e from the necessary conditions for a minimum Trace P_eQ_e.

6) Use the partial derivative equations with respect to the free parameter matrices to calculate gradient directions.

7) Set $P_i^j + 1 = \Delta P_i + P_i^j$ where $\Delta P_i = -s\partial J/\partial P_i$, $i = 1, \ldots, k$.

8) Repeat until P_i, $i = 1, \ldots, k$ converge to their optimal values.

9) Use calculated P_i, $i = 1, \ldots, k$, to determine controller parameters based on the parameterization given by (3.100) to complete the controller design.

It is noted that in view of (3.109)-(3.113) the last two necessary conditions in (3.115) are linear in P_1, P_2. However, they represent coupled Sylvester equations, and so do not reduce to a computationally attractive Riccati-based algorithm. In the centralized case however, one can use the Riccati-based algorithm provided $(CGG^TC^T)^{-1}$ exits.

3.5.4 Example

In order to illustrate the approach, consider a seventh-order system with two decentralized dynamic controllers to be designed so as to minimize the effects of a disturbance input on the regulated outputs. A system of the form (3.93) will be used, characterized by the following matrices:

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & -1 & 1 & 0 \\ -2 & -3 & 1 & 0 & 0 & 1 & 1 \\ -2 & -3 & -2 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & -1 & -2 & 1 & 1 \\ -1 & 0 & -3 & 0 & -2 & -2 & -4 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = \operatorname{diag}\{100, 10, 100, 0, 0, 0, 0\}, \quad R_{11} = R_{22} = 1.$$

The open-loop system is unstable, with the spectrum,

$$\Lambda(A) = \{-2.80 \pm j2.41, -4.48, -3.71, .83 - .68, -1.26\}.$$

The reference system (here an LQ solution, i.e., $\gamma \to \infty$) is characterized by the optimal spectrum,

$$\Lambda(F) = \{-10.09, -1.62 \pm j1.63, -1.15, -4.5, -3.10 \pm j.87\}.$$

Since in this problem $r_1 = r_2 = 2$, two modes of the reference solution can be retained with static projective controls. In addition, by using two first-order dynamic controllers, one additional mode can be retained. The dominant modes are chosen for retention; thus,

$$\Lambda_r = \begin{bmatrix} -1.62 + j1.63 & 0 \\ 0 & -1.62 - j1.63 \end{bmatrix} \quad \Lambda_p = [-1.15].$$

The initial choices for P_1 and P_2 are

$$P_1 = [0 \quad 0], \quad P_2 = [0 \quad 0]$$

producing the closed-loop spectrum,

$$\Lambda(\tilde{A}_{ce}) = \{-3.65 \pm j2.16, -3.89, \ .36, -1.62 \pm j1.64, -.92, -1.15, -1.15\}.$$

Note that this choice of P_1 and P_2 fails to stabilize the resulting closed loop system. Consequently, the embedding parameter method must be used initially until a stable A_{ce} is achieved, or it is determined that the order of the dynamic controllers must be increased. For the example at hand, first-order controllers did produce a stable system; thus, the order did not need to be increased.

The feasible direction algorithm is then employed to yield the optimum parameters for P_1 and P_2 for disturbance minimization. These optimal values are found to be

$$\begin{aligned} P_1 &= [-2.41 \quad 0.71] \\ P_2 &= [-3.65 \quad 1.58] \end{aligned}$$

with an optimal value of the cost criterion of

$$J = \text{Trace } P_e Q_e = 1.430E^{-2}.$$

(Notice that for the initial choice, $J = \infty$, since the system was unstable.) The closed loop spectrum, $\tilde{A}_{ce}$, produced by the free parameter matrices is now

$$\Lambda(\tilde{A}_{ce}) = \{-5.77 \pm 3.36j, -.58 \pm 1.72j, -4.26, -3.42, -1.62 \pm 1.63j, -1.15\}.$$

Once P_1 and P_2 are determined, the controller parameters can then be determined from (3.100) to be

$$\begin{array}{rclrl} H_1 & = & -1.93, & H_2 = -7.92 \\ D_1 & = & [-5.48 \quad -4.58], & D_2 = [-25.17 \quad 17.05] \\ K_{d1} & = & [-3.43 \quad -1.95], & K_{d2} = [-34.17 \quad 20.48] \\ N_{d1} & = & -0.95 & N_{d2} = -8.87, \end{array}$$

thus completing the design.

3.6 A design example

We finally present a realistic design example to illustrate the design procedures developed in these two sections. The structure considered is a 45 foot lattice-type, light-weight (5 lbs.), flexible beam with fixed base and free tip shown in Figure 3.5. The system is modeled by a 40th order state space model.

The control $u \in \mathbb{R}^2$ consists of torques applied at the base of the structure about the x and y axes. The disturbance $w \in \mathbb{R}^2$ is generated by an $x - y$ translation applied to the base where the z-axis is taken to be the axis of the cruciform. Measurements of the system $y \in \mathbb{R}^8$ are obtained from an $x - y$ axis gyro and accelerometer sensors located at the tip and base of the structure. The controlled output $z \in \mathbb{R}^4$ is the position measurement at the tip and base of the structure. The model parameters can be found in Figure 3.6 to 3.8 [35]. This example has also been studied in [36] and [37].

The system considered in this example consists of the x-axis dynamics of the complete system. The inputs and outputs were decentralized after a Generalized Hessenberg analysis [37] of the system. The resulting systems are described in Table 3.1. For more details see [35], [37].

The 12th order model of the x-axis dynamics was obtained by performing a balanced reduction on the original system using only the x-axis inputs and outputs. The eigenvalues

Table 3.1: Modes of the Cruciform Model.

Real	Imaginary	Frequency	Damping
-4.2471e-03	±8.4941e-01	8.4942e-01	5.0000e-03
-3.4723e-02	±7.0339e+00	7.0340e+00	4.9365e-03
-3.0800e-02	±7.7296e+00	7.7297e+00	3.9846e-03
-1.9863e-01	±1.0458e+01	1.0460e+01	1.8989e-02
-5.1863e-01	±2.5925e+01	2.5930e+01	2.0001e-02
-8.2976e-01	±4.6092e+01	4.6100e+01	1.7999e-02

of the resulting model are found in Table 3.1 and show typical flexible structure properties, i.e., lightly damped and closely packed low frequency modes. The y-axis dynamics were treated similarly but are not shown here.

Note that model reduction was used only to remove very weakly controllable and observable modes. This is done to avoid neglecting modes which may be important in the design of the controller since the projective controls method allows one to use a high order model without requiring a high order controller.

The cruciform model is decentralized into x-axis and y-axis systems as shown in Table 3.2.

Table 3.2: GHR Decentralized Model Results.

Model	Inputs	Outputs
x-axis	u_1, w_2	$z_2, z_5, y_1, y_4, y_8, y_{11}$
y-axis	u_2, w_1	$z_1, z_4, y_2, y_5, y_7, y_{10}$

3.6.1 Design of the controller.

A projective controller is now designed to achieve disturbance attenuation for the cruciform system. From Figure 3.9 it can be seen that the nominal system has disturbance attenuation of $\gamma = -20$dB. In this example, the disturbance attenuation will be improved to $\gamma = -40$dB using a low-order, robust controller designed using the projective controls method.

A reference a state feedback controlled system was determined first. The eigenvalues of the resulting closed loop system, $\lambda(F)$ are given in Table 3.3. The frequency response of the

Table 3.3: Modes of the Reference System.

	Real	Imaginary	Frequency	Damping
a	-1.0856e-01	±8.5272e-01	8.5960e-01	1.2629e-01
b	-3.4723e-02	±7.0339e+00	7.0340e+00	4.9365e-03
c	-3.0801e-02	±7.7296e+00	7.7297e+00	3.9848e-03
d	-1.9902e-01	±1.0458e+01	1.0460e+01	1.9026e-02
e	-5.1870e-01	±2.5925e+01	2.5930e+01	2.0004e-02
f	-8.2977e-01	±4.6092e+01	4.6100e+01	1.7999e-02

closed-loop system in Figure 3.10 shows the disturbance attenuation to be $\gamma = -40$dB. The gain margins are $(0, \infty)$ and phase margins are $\pm 90°$. Thus, this state feedback forms an acceptable reference system for the projective controls method.

Static projective controls excited the higher frequency models and so second order ($p = 2$) dynamic projective controller was considered. In the design of the dynamic strictly proper projective controller, there are two main design freedoms: the selection of the retained modes $[\lambda_p, X_p]$ and the selection of the design parameter P_o. The retained modes $[\lambda_p, X_p]$ are chosen to retain disturbance attenuation and robustness properties as much as possible. The mode a is retained in order to preserve the damping of this mode for disturbance attenuation. Thus, $\lambda_p = \{a\}$.

To select P_o, the approach of Section 3.4 was applied. Using a gradient method, the Frobenius-Hankel norm of E_2 was minimized. The frequency response of the dynamic controller associated with this choice of D_o is given in Figure 3.11.

3.6.2 Evaluation of Design

The final step was to evaluate the design by applying the controller to the full system. The spectrum of the resulting closed loop system is shown in Table 3.4. The disturbance attenuation of the full system is $\gamma = -40$dB as seen in Fig. 3.13. Thus, the disturbance attenuation goals of the design have been met using a second order controller. The gain margins are $[0, 40dB]$ and phase margins are $\pm 70°$ which approach the stability margins of the state feedback system. However, if these results are not satisfactory, a higher order controller could be considered.

To demonstrate the disturbance attenuation achieved by this design in the time domain, the time response of the system to a disturbance impulse is computed. The open loop response is given in Figure 3.14. Note the low damping of the low frequency mode. For the system controlled by the design given above, the response is given in Figure 3.15. In this case, the damping on the low frequency mode has increased dramatically.

A controller was also designed for the y-axis dynamics of the system in a similar manner. The resulting closed loop system with decentralized controls was seen to be stable and retain the desired disturbance attenuation properties. Thus, the disturbance attenuation of an flexible system has been improved using two second order, decentralized controllers designed by the method of this paper.

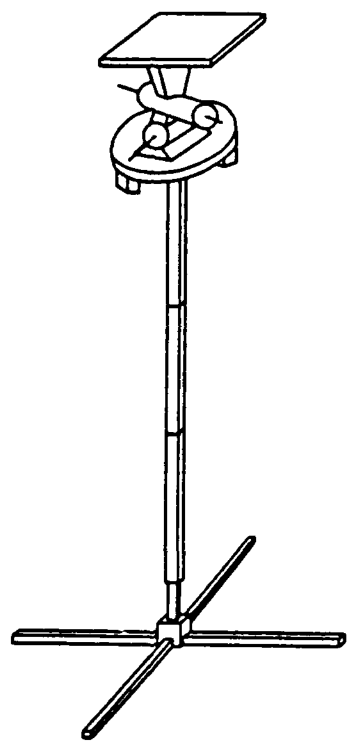

Figure 3.5: The Cruciform Structure.

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Dw \\
y &= Cx + Gu + Hw \\
z &= Ex
\end{aligned}
$$

$$
\begin{aligned}
A &= \begin{bmatrix} -2\xi\Omega & -\Omega^2 \\ I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_L \\ 0 \end{bmatrix}, \\
G &= \begin{bmatrix} D_L \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_r & 0 \\ -2C_a\xi\Omega & -C_a\Omega^2 \end{bmatrix}, \\
F &= \begin{bmatrix} 0 \\ C_aB_L \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ C_aD_L \end{bmatrix}, \\
H &= \begin{bmatrix} 0 & C_a \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega &= \text{diag} \begin{bmatrix} .00025 & .00001 & .00083 & .84942 & .87782 & 2.2420 & 7.0290 & 7.2100 & 7.3300 & 7.6228 \\ 7.7327 & 7.9790 & 8.0682 & 8.5011 & 10.460 & 19.910 & 20.380 & 25.930 & 42.760 & 46.100 \end{bmatrix}, \\
\xi &= \text{diag} \begin{bmatrix} .000 & .000 & .000 & .005 & .020 & .005 & .005 & .005 & .005 & .005 \\ .005 & .020 & .005 & .016 & .019 & .020 & .005 & .020 & .011 & .018 \end{bmatrix},
\end{aligned}
$$

Figure 3.6: Model Structure.

$$
B_L = \begin{bmatrix}
7.144e-07 & 8.829e-13 \\
-6.267e-14 & -2.170e-08 \\
-1.651e-11 & 1.380e-11 \\
-1.564e-02 & 2.570e-07 \\
3.654e-07 & 1.385e-02 \\
1.043e-04 & -1.007e-04 \\
1.770e-02 & 4.871e-04 \\
-2.549e-03 & -3.550e-02 \\
-2.355e-03 & 3.588e-02 \\
1.184e-02 & -2.322e-02 \\
-1.925e-02 & -7.041e-03 \\
-5.141e-03 & -7.797e-02 \\
1.080e-02 & -2.006e-02 \\
-5.175e-04 & 1.243e-01 \\
-1.916e-01 & -2.810e-05 \\
-7.178e-05 & 1.127e-01 \\
-1.459e-03 & -5.674e-03 \\
2.716e-01 & -2.096e-06 \\
-9.120e-06 & -4.698e-02 \\
-1.770e-01 & 2.299e-06
\end{bmatrix}, \quad
D_L = \begin{bmatrix}
1.417e-09 & 3.503e-02 \\
3.807e-02 & -6.824e-11 \\
-3.549e-07 & -1.698e-08 \\
-1.522e-07 & -6.110e-03 \\
-7.738e-03 & 1.342e-07 \\
1.393e-05 & 7.851e-06 \\
-3.123e-05 & 4.292e-04 \\
2.258e-03 & -5.985e-05 \\
-2.273e-03 & -5.487e-05 \\
1.449e-03 & 2.707e-04 \\
4.377e-04 & -4.396e-04 \\
4.806e-03 & -1.141e-04 \\
1.234e-03 & 2.386e-04 \\
-7.523e-03 & -1.114e-05 \\
1.616e-06 & -3.771e-03 \\
-5.869e-03 & -1.181e-06 \\
2.946e-04 & -2.389e-05 \\
1.052e-07 & 4.242e-03 \\
2.175e-03 & -1.293e-07 \\
-1.050e-07 & -2.473e-03
\end{bmatrix},
$$

Figure 3.7: Model Input Parameters.

$$C_r^T = \begin{bmatrix}
7.144e-07 & 0.000e+00 & -3.946e-06 & 9.562e-07 & 0.000e+00 & -3.949e-06 \\
0.000e+00 & -2.170e-08 & -4.200e-06 & 0.000e+00 & -5.436e-08 & -4.200e-06 \\
0.000e+00 & 0.000e+00 & 2.224e-01 & 0.000e+00 & 0.000e+00 & 2.224e-01 \\
-1.564e-02 & 2.570e-07 & -2.994e-04 & -1.710e-02 & 3.323e-07 & 2.358e-04 \\
3.654e-07 & 1.386e-02 & -3.112e-04 & 3.967e-07 & 1.778e-02 & 2.829e-04 \\
1.043e-04 & -1.007e-04 & -6.333e-02 & 5.561e-06 & -6.236e-06 & 7.231e-01 \\
1.770e-02 & 4.871e-04 & 5.976e-06 & -2.657e-02 & -4.646e-04 & -7.021e-04 \\
-2.346e-03 & -3.560e-02 & 3.897e-04 & 2.387e-03 & 3.479e-02 & -4.811e-02 \\
-2.386e-03 & [illegible]-02 & 1.223e-04 & 2.399e-03 & -3.917e-02 & -1.556e-02 \\
1.184e-02 & -2.372e-02 & 3.809e-04 & -1.353e-02 & 1.884e-02 & -5.238e-02 \\
-1.926e-02 & -7.041e-03 & 2.500e-06 & 2.752e-02 & 5.873e-03 & -3.534e-03 \\
-6.141e-03 & -7.797e-02 & -2.456e-04 & 4.962e-03 & 6.735e-02 & 3.689e-02 \\
1.080e-02 & -3.006e-02 & 3.140e-04 & -1.078e-02 & 1.836e-02 & -4.819e-02 \\
-6.175e-04 & 1.243e-01 & 3.521e-06 & 5.377e-04 & -6.600e-02 & -5.972e-03 \\
-1.916e-01 & -2.801e-06 & 3.690e-05 & 1.823e-01 & 7.747e-06 & -9.256e-03 \\
-7.178e-06 & 1.127e-01 & -1.329e-04 & -3.891e-06 & 2.829e-01 & 9.886e-02 \\
-1.489e-03 & -5.674e-03 & -2.518e-03 & -8.886e-04 & -1.464e-02 & 1.937e+00 \\
2.716e-01 & -2.096e-06 & -9.656e-06 & 2.962e-01 & -5.066e-06 & 9.887e-03 \\
-9.120e-06 & -4.698e-02 & 7.325e-06 & 1.072e-06 & 3.637e-01 & -6.133e-03 \\
-1.770e-01 & 2.299e-06 & -9.101e-06 & 3.388e-01 & -2.373e-05 & 5.256e-03
\end{bmatrix}$$

$$C_e^T = \begin{bmatrix}
1.417e-09 & 3.503e-02 & 3.468e-23 & 1.432e-09 & 3.502e-02 & 4.663e-17 \\
3.807e-02 & 0.000e+00 & -1.463e-24 & 3.807e-02 & 0.000e+00 & -2.187e-18 \\
-3.549e-07 & -1.698e-08 & -1.329e-26 & -3.546e-07 & -1.673e-08 & 1.440e-21 \\
-1.572e-07 & -6.110e-03 & 2.384e-15 & 4.119e-06 & 2.222e-01 & 3.055e-09 \\
-7.738e-03 & 1.342e-07 & -2.582e-15 & 2.218e-01 & -5.184e-06 & -3.310e-09 \\
1.393e-05 & 7.851e-06 & 4.611e-18 & -1.267e-03 & -1.195e-03 & 5.910e-12 \\
-3.123e-05 & 4.292e-04 & 1.387e-10 & 7.993e-05 & 1.406e-02 & 1.778e-04 \\
2.258e-03 & -5.985e-05 & -3.414e-11 & 7.045e-03 & 4.497e-03 & -4.376e-05 \\
-2.273e-03 & -5.487e-05 & 1.290e-10 & -3.180e-02 & 3.039e-03 & 1.653e-04 \\
1.449e-03 & 2.707e-04 & 9.995e-12 & -1.190e-03 & -6.003e-03 & 1.274e-05 \\
4.377e-04 & -4.396e-04 & 2.020e-10 & 1.456e-03 & -1.779e-02 & 2.588e-04 \\
4.806e-03 & -1.141e-04 & 1.740e-10 & 5.317e-02 & 5.608e-03 & 2.230e-04 \\
1.234e-03 & 2.388e-04 & 8.846e-11 & 2.096e-02 & -9.427e-03 & 1.134e-04 \\
-7.523e-03 & -1.114e-05 & 9.249e-11 & -7.424e-02 & 1.910e-04 & 1.185e-04 \\
1.616e-06 & -3.771e-03 & -1.110e-11 & 4.036e-05 & -1.311e-01 & -1.423e-05 \\
-5.869e-03 & -1.181e-06 & -7.828e-12 & 1.680e-01 & -2.111e-06 & -1.002e-05 \\
2.946e-04 & -2.389e-05 & 4.332e-13 & -9.449e-03 & -3.073e-04 & 5.543e-07 \\
1.052e-07 & 4.242e-03 & -1.060e-11 & -6.058e-06 & -1.686e-01 & -1.343e-05 \\
7.175e-03 & -1.293e-07 & -5.344e-11 & 1.906e-01 & -3.141e-06 & -6.801e-05 \\
-1.050e-07 & -2.473e-03 & -8.271e-11 & -1.476e-05 & -1.763e-01 & -1.051e-04
\end{bmatrix}$$

Figure 3.8: Model Output Parameters.

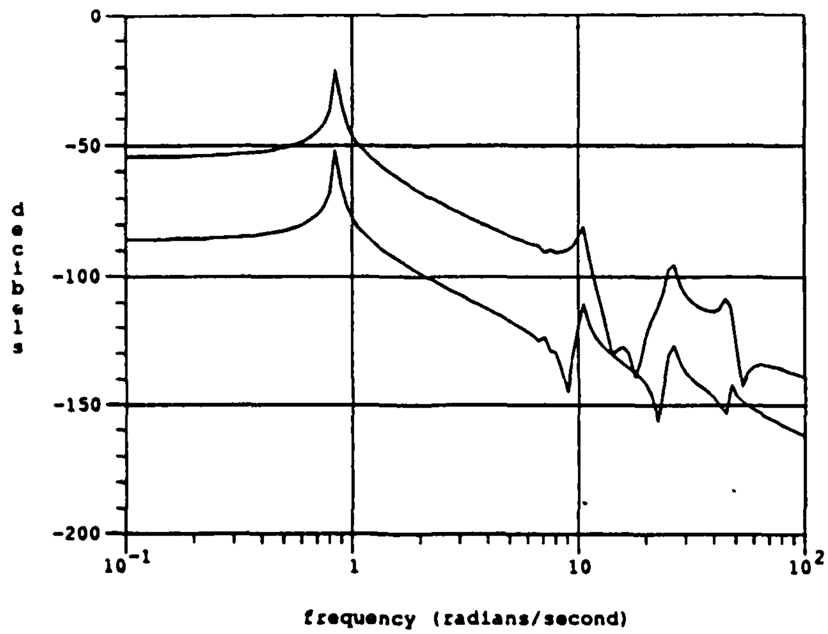

Figure 3.9: Frequency Response of the Open-Loop System.

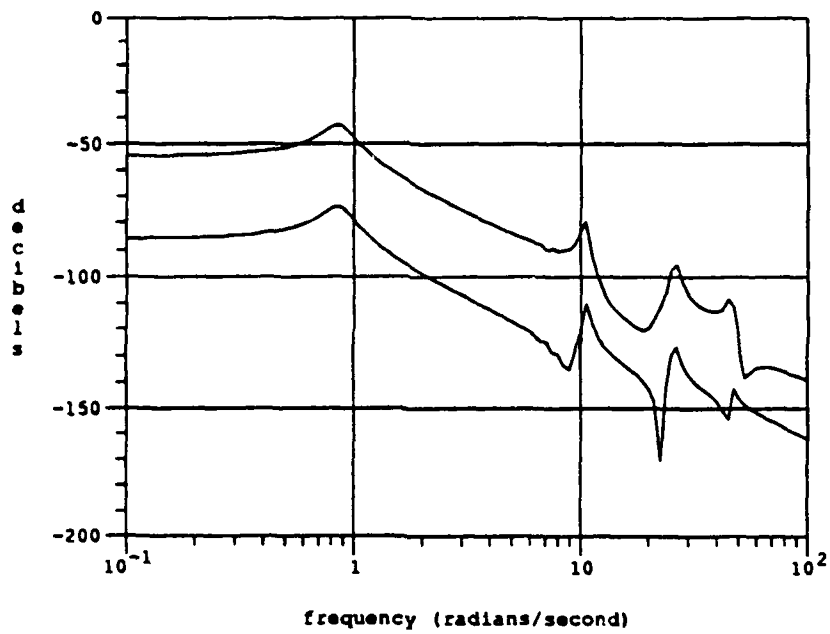

Figure 3.10: Frequency Response of the Reference System.

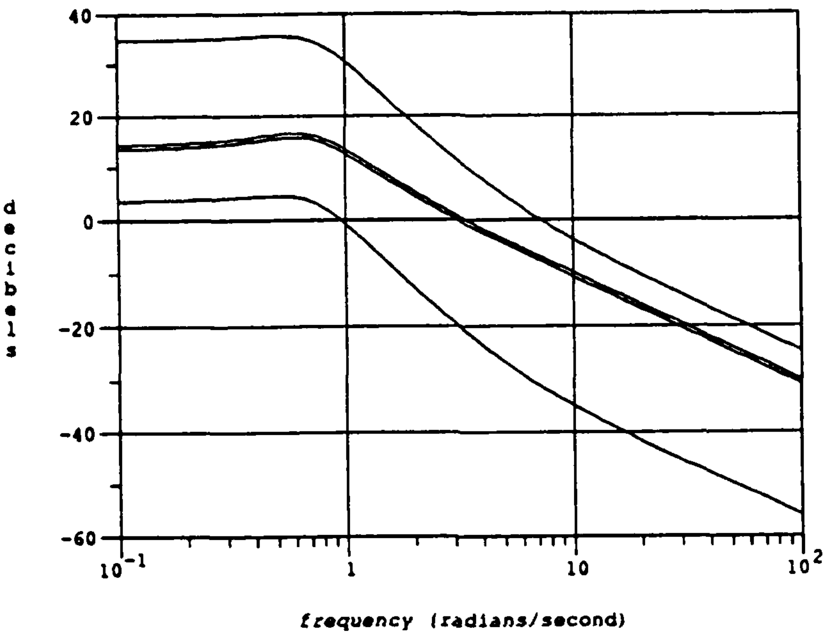

Figure 3.11: Frequency Response of the Compensator.

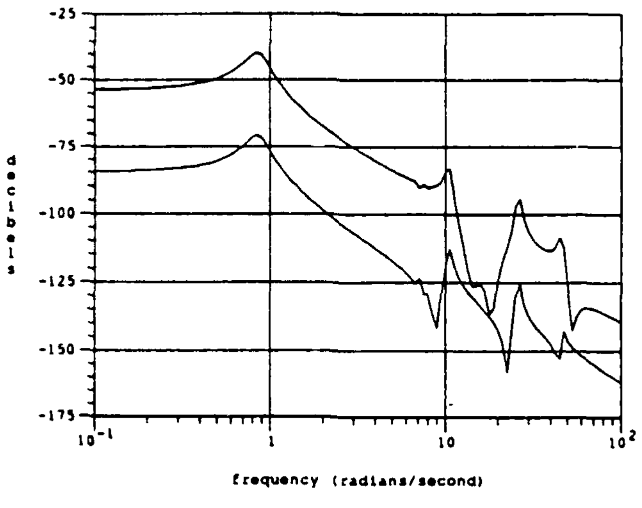

Figure 3.12: Frequency Response of the Closed-Loop System.

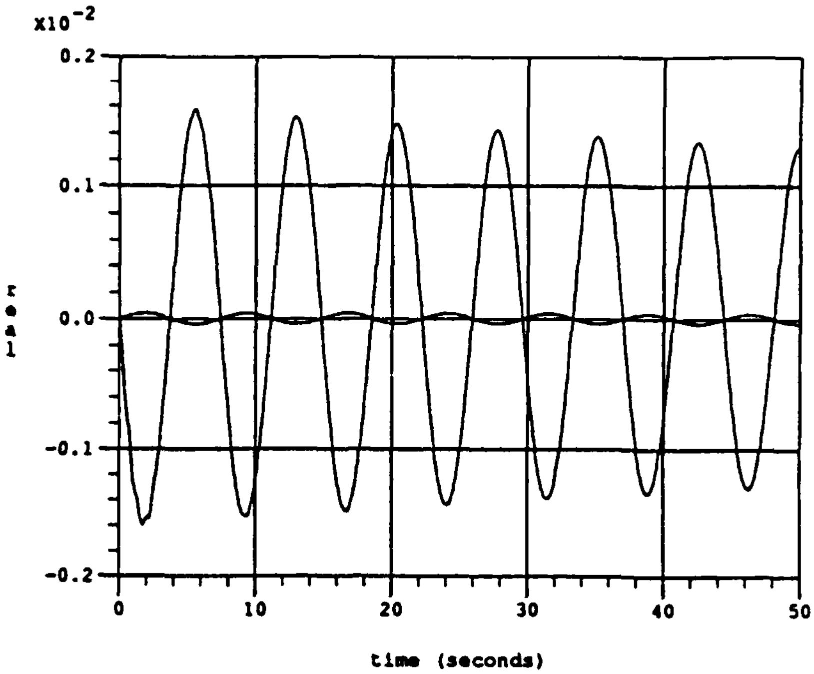

Figure 3.13: Time Response of the Open-Loop System.

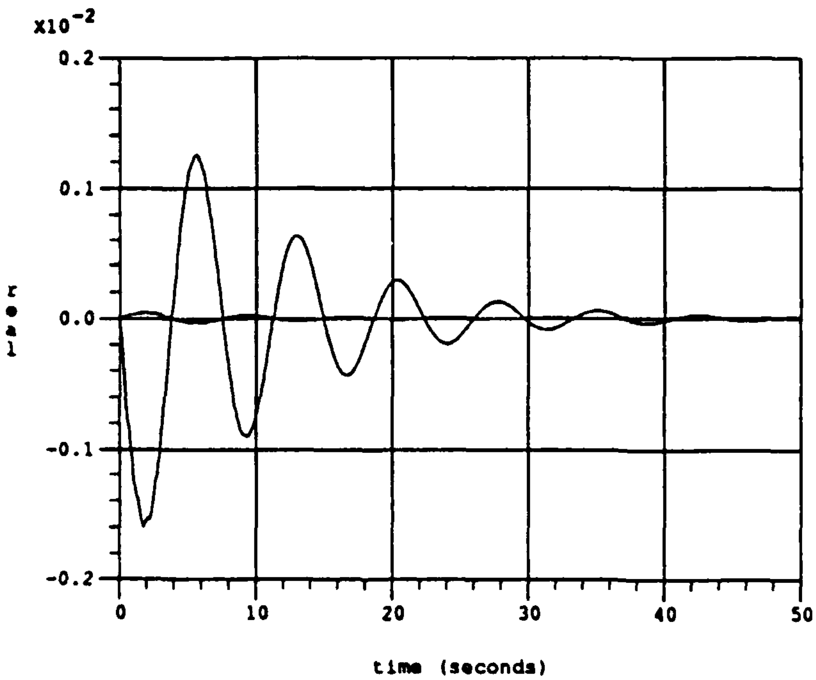

Figure 3.14: Time Response of the Closed-Loop System.

4 H_∞ DISTURBANCE REJECTION VIA THE ALGEBRAIC RICCATI EQUATION

4.1 Introduction

Sections 2 and 3 have concentrated on results useful in developing methodologies for low-order controller design. For example, the projective controls approach allows a convenient parameterization of low-order controllers which retain some spectral characteristics of a desirable reference controller, and the FH-norm minimization approach provides a means of determining these free parameters to guarantee-disturbance rejection for the closed-loop system. The determination of the free parameters of a low-order controller so as to directly minimize or bound the H_∞ norm of the closed-loop system is an open issue. While necessary conditions can be derived, see for example [38,39,40,41], the existence issue is open, and moreover, convergence of available algorithms is not guaranteed even if a solution exists.

A great deal of research has been conducted on H_∞ control design and the related issue of robust control. For a survey of these two areas, see [42] and its references. In the last few years, the connection between H_∞ control and the algebraic Riccati equation (ARE) has been established. See, for example, [43], [44], [2], [45], [46], [47], [48], [49], and [1].

This section concentrates on the ARE based methodology for controller design by which minimizing, or bounding, the H_∞ norm of the closed-loop system is accomplished. The methodology is useful for deriving three classes of controllers: (i) state-feedback controllers; (ii) full-order observer-based output-feedback controllers; and (iii) decentralized controllers comprising a full-order observer of the plant in each control channel. The state-feedback and centralized output-feedback control laws derived by this methodology have been developed in previous work, such as [48], [1], [47], and [46]. However, the decentralized control laws are new, and represent a novel approach to H_∞-norm-bounding decentralized control design.

For convenience, we restate the standard disturbance-rejection problem to be addressed. Consider a linear, time-invariant plant of the form

$$\dot{x} = Ax + Bu + Gw_0, \tag{4.1a}$$

$$y = Cx + w, \tag{4.1b}$$

$$z = \begin{pmatrix} Hx \\ u \end{pmatrix}, \tag{4.1c}$$

where x is the state of the plant, u is the control input, y is a measured output, z is an output to be regulated, and w_0 and w_1 are square-integrable disturbances. Note that (4.1) is essentially the same as (2.1), with $D = 0$.

Given any control input u to the plant (4.1), define the cost functional

$$J(u) = \sup\left\{ \frac{||z||_2}{||w_0||_2} : w_0 \in L_2[0,\infty), w_0 \neq 0 \right\}.$$

Note that the measurement (4.1b), and hence the measurement noise w, is not considered in this definition. Therefore, the cost $J(u)$ is associated with open-loop controls or state-feedback controls. In the case where u is a state-feedback control, $J(u)$ is the H_∞ norm of the closed-loop transfer-function matrix from w_0 to z. Define the optimal cost as

$$\alpha_\infty = \inf\{J(u) : u \in L_2[0,\infty)\}. \tag{4.2}$$

The following theorem from [48] gives a means of determining α_∞, and also establishes that there exists a state-feedback control law which achieves any H_∞-norm bound larger than α_∞.

Theorem 4.1. *For the plant (4.1) with (A, B) stabilizable, and (A, H) detectable, the bound*

$$\alpha_\infty < \alpha$$

holds if and only if

$$A^TX + XA + \frac{1}{\alpha^2}XGG^TX - XBB^TX + H^TH = 0, \tag{4.3}$$

with $X \geq 0$ and $A_\alpha \equiv A - BB^TX + \alpha^{-2}GG^TX$ Hurwitz. If so, the state-feedback control law

$$u = -B^TXx \tag{4.4}$$

stabilizes the plant, and gives a closed-loop transfer-function matrix

$$T(s) = \begin{pmatrix} H \\ -B^TX \end{pmatrix}(sI - A + BB^TX)^{-1}G$$

from w_0 to z satisfying $||T||_\infty < \alpha$.

If the control u for plant (4.1) must be generated by a controller that uses only the measurement y given by (4.1b), then the relevant cost functional is

$$J_0(u) = \sup\left\{\frac{||z||_2}{||w_e||_2}; w_e \in L_2[0,\infty), w_e^T = [w_o^T \;\; w^T] \neq 0\right\}.$$

The infimal value of the output-feedback cost $J_0(u)$ is generically greater than α_∞ defined in (4.2). The following theorem from [48] or [1] gives a means of determining this greatest lower bound, and also gives an output-feedback control law which guarantees any given H_∞-norm bound achievable by output feedback.

Theorem 4.2. *In the plant (4.1), assume (A, B) stabilizable, (A, C) detectable, (A, G) stabilizable, and (A, H) detectable. Then there exists a stabilizing controller such that the closed-loop transfer-function matrix $T(s)$ from w_e to z satisfies $||T||_\infty < \alpha$ if and only if*

$$A^TX + XA + \frac{1}{\alpha^2}XGG^TX - XBB^TX + H^TH = 0 \tag{4.3}$$

with $X \geq 0$ and $A_\alpha \equiv A - BB^TX + \alpha^{-2}GG^TX$ Hurwitz,

$$AY + YA^T + \frac{1}{\alpha^2}YH^THY - YC^TCY + GG^T = 0 \tag{4.5}$$

with $Y \geq 0$ and $A - YC^TC + \alpha^{-2}YH^TH$ Hurwitz, and

$$\sigma_{\max}\{YX\} < \alpha^2. \tag{4.6}$$

If so, then the output-feedback control law

$$\dot{\xi} = (A + \frac{1}{\alpha^2}GG^TX - BB^TX - LC)\xi + Ly, \tag{4.7a}$$

$$u = -B^TX\xi \tag{4.7b}$$

with

$$L = (I - \alpha^{-2}YX)^{-1}YC^T \tag{4.8}$$

stabilizes the plant and gives a closed-loop transfer-function matrix from w_e to z satisfying $||T||_\infty < \alpha$.

If a decentralized control structure is imposed or desired for a given problem, then the result given in Theorem 4.2 cannot be used. A great deal of attention has been paid to the problem of decentralized control design; see, for example, [50], [51], [52], [38], [53], [54], [55], and [56]. Unlike previous work, the decentralized design procedure derived here addresses the issue of H_∞ suboptimal control in a decentralized setting, and results in observer-based designs that rely on a known state-feedback H_∞-norm-bounding control. The observer structure assumed for the controllers allows the derivation of two design equations for the decentralized control law: One of these is the standard state-feedback H_∞ design equation; the other is a Riccati-like algebraic equation (RLAE).

The decentralized-control version of the disturbance-rejection problem is formulated and a solution derived in Section 4.5. The approach is based on a fundamental lemma described next, and the spirit of the approach is then illustrated on the centralized control problem where the result in Theorem 4.2 is rederived.

4.2 The Key Lemma

The following lemma establishes a sufficient condition, in the form of an "algebraic Riccati inequality," for a given system to be stable and have a particular H_∞-norm bound. This condition provides the basis for all the control laws derived in the remainder of Section 4, as well as those derived in Section 5. The lemma is a simple extension of Lemma 1 of [57].

Lemma 4.1. *Let $T(s) = H(sI - F)^{-1}G$, with (F, H) a detectable pair. If there exist a real matrix $X \geq 0$ and a positive scalar γ such that*

$$F^T X + XF + \frac{1}{\gamma^2} XGG^T X + H^T H \leq 0, \tag{4.9}$$

then F is Hurwitz, and $T(s)$ satisfies

$$||T||_\infty \leq \gamma. \tag{4.10}$$

Proof. Suppose (4.9) holds, with $X \geq 0$. To show that F is Hurwitz, let $v \neq 0$ satisfy

$$Fv = \lambda v.$$

Multiply (4.9) on the left by v^* and on the right by v to obtain

$$2Re(\lambda)v^*Xv + \frac{1}{\gamma^2}v^*XGG^TXv + v^*H^THv \leq 0. \tag{4.11}$$

Now, $2Re(\lambda)v^*Xv \leq 0$ since all other terms on the left-hand side of (4.11) are non-negative. If $Re(\lambda)v^*Xv < 0$, then $v^*Xv > 0$ and $Re(\lambda) < 0$. If, on the other hand, $Re(\lambda)v^*Xv = 0$, then all terms in (4.11) must be zero. Therefore, the eigenvector v of F is in the null space of H. Since (F, H) is detectable, the corresponding eigenvalue must be in the open left-half plane. In either case, $Re(\lambda) < 0$; thus, F is Hurwitz.

Now, to prove (4.10), let $\omega \in \mathbb{R}$; add and subtract $j\omega X$ to obtain from (4.9)

$$-(-j\omega I - F^T)X - X(j\omega I - F) + \frac{1}{\gamma^2}XGG^TX + H^TH \leq 0. \tag{4.12}$$

Since F is Hurwitz, $(j\omega I - F)$ is invertible. Define

$$K(j\omega) = \frac{1}{\gamma^2}G^TX(j\omega I - F)^{-1}G;$$

pre-multiply (4.12) by $\frac{1}{\gamma}G^T(-j\omega I - F^T)^{-1}$, and post-multiply by $\frac{1}{\gamma}(j\omega I - F)^{-1}G$ to obtain

$$-K(j\omega) - K^T(-j\omega) + K^T(-j\omega)K(j\omega) + \frac{1}{\gamma^2}T^T(-j\omega)T(j\omega) \leq 0,$$

which gives

$$I - \frac{1}{\gamma^2}T^T(-j\omega)T(j\omega) \geq [I - K^T(-j\omega)][I - K(j\omega)].$$

Therefore, for all $\omega \in \mathbb{R}$,

$$I - \frac{1}{\gamma^2}T^*(j\omega)T(j\omega) \geq [I - K(j\omega)]^*[I - K(j\omega)] \geq 0,$$

which implies (4.10). □

4.3 The General Approach

Lemma 4.1 suggests an approach to deriving H_∞-norm-bounding control designs: The approach is to first fix a controller structure, so as to determine the form of the closed-loop system

$$\dot{x}_e = F_e x_e + G_e w_e, \quad z = H_e x_e, \tag{4.13}$$

and then select feedback and observer gains so that the algebraic Riccati equation

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e = 0 \tag{4.14}$$

has a solution $X_e \geq 0$. By Lemma 4.1, if (F_e, H_e) is a detectable pair, then the closed-loop system (4.13) is stable, and $T(s) = H_e(sI - F_e)^{-1}G_e$ satisfies $\|T\|_\infty \leq \alpha$.

By taking this approach, the same state-feedback and output-feedback control designs given in [48] and [1] according to Theorems 4.1 and 4.2 are recovered in Section 4.4. These derivations are simple, and serve to illustrate the approach and to introduce the derivation of a new observer-based decentralized control law in Section 4.5. In the decentralized case, controller feedback gains are computed from the solution to a state-feedback design ARE, while observer gains are computed from a Riccati-like algebraic equation. The existence of appropriate solutions to the design equations is sufficient to guarantee the control to be H_∞-norm-bounding.

4.4 The Centralized Control Design

The output-feedback H_∞ control law given in Theorems 4.1 and 4.2 are now derived, based on Lemma 4.1. This derivation, which also appears in [11], is not a complete proof of Theorems 4.1 and 4.2, in that it establishes only that the designs are sufficient to guarantee a predetermined H_∞-norm bound, and not that any achievable bound can be obtained using such designs. For this reason, not all the stabilizability and detectability conditions appearing in Theorem 4.1 and 4.2 are needed.

The problem here is to derive control laws to stabilize the plant (4.1) and provide an H_∞-norm bound for the closed-loop transfer function matrix from the disturbance $w_e = \binom{w_0}{w}$ to

z. By Lemma 4.1, a sufficient condition for a state-feedback control $u = Kx$ to stabilize the plant and guarantee the H_∞-norm bound $\|T\|_\infty \leq \alpha$ is that the feedback gain matrix K satisfy

$$(A+BK)^T X + X(A+BK) + \frac{1}{\alpha^2} XGG^T X + (H^T \; K^T)\begin{pmatrix} H \\ K \end{pmatrix} = 0 \tag{4.15}$$

with $X \geq 0$. Rearrange (4.15) as

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XBB^T X + (K^T + XB)(K + B^T X) + H^T H = 0,$$

which, upon setting $K = -B^T X$, gives the state-feedback design equation (4.3). If (A, H) is a detectable pair, then the detectability condition of Lemma 4.1 is satisfied for the closed-loop system. Thus, the state-feedback design given in Theorem 4.1 is recovered via Lemma 4.1.

In the output-feedback case, an observer-based control law will be used to approximate a state-feedback control $u = Kx$. To mimic the dynamics of the plant (4.1), the observer takes the form

$$\dot{\xi} = A\xi + Bu + G\hat{w}_0 + L(y - C\xi), \quad u = K\xi, \tag{4.16a}$$

where a state-feedback model of the disturbance w_0 is assumed as

$$\hat{w}_0 = K_d \xi. \tag{4.16b}$$

The feedback gain K, observer gain L, and disturbance-estimate gain K_d will be chosen so that, when controller (4.16) is applied to the plant (4.1), the closed-loop system will satisfy the hypotheses of Lemma 4.1.

Introduce the error vector $e = \xi - x$, and write the closed-loop system as

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A+BK & BK \\ GK_d & A+GK_d-LC \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} G & 0 \\ -G & L \end{pmatrix}\begin{pmatrix} w_0 \\ w \end{pmatrix} \equiv \tilde{F}_e x_e + \tilde{G}_e w_e, \tag{4.17a}$$

$$z = \begin{pmatrix} H & 0 \\ K & K \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} \equiv \tilde{H}_e x_e. \tag{4.17b}$$

Similar to the state-feedback case, the goal is to find $\tilde{X}_e \geq 0$ such that

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2}\tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.18}$$

To ensure decoupling of (4.18) into a state-feedback design ARE and an observer design equation, look for a block-diagonal solution

$$\tilde{X}_e = \begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix} \geq 0.$$

Then, the upper-left block of (4.18) is exactly Equation (4.15). If, as in the state-feedback solution, $X \geq 0$ solves (4.3) and the feedback gain is given by

$$K = -B^T X, \tag{4.19}$$

then the upper-left block of (4.18) is satisfied. The upper-right block of (4.18) then gives

$$-XBB^TX + K_d^T G^T X_1 - \frac{1}{\alpha^2} XGG^T X_1 + XBB^T X = 0,$$

which is satisfied if

$$K_d = \frac{1}{\alpha^2} G^T X. \tag{4.20}$$

Given the choices (4.19) and (4.20), the lower-right block of (4.18) gives

$$\begin{aligned} X_1(A + \alpha^{-2}GG^TX - LC) &+ (A + \alpha^{-2}GG^TX - LC)^T X_1 \\ &+ \frac{1}{\alpha^2} X_1(GG^T + LL^T)X_1 + XBB^TX = 0. \end{aligned} \tag{4.21}$$

Add to (4.21) the design equation (4.3) to obtain the ARE

$$\begin{aligned} (X + X_1)A &+ A^T(X + X_1) + \frac{1}{\alpha^2}(X + X_1)GG^T(X + X_1) - \alpha^2 C^T C + H^T H \\ &+ \left(\frac{1}{\alpha} X_1 L - \alpha C^T\right)\left(\frac{1}{\alpha} L^T X_1 - \alpha C\right) = 0, \end{aligned} \tag{4.22}$$

which suggests the choice for the observer gain L as

$$X_1 L = \alpha^2 C^T. \tag{4.23}$$

In order that L satisfying (4.23) is guaranteed to exist, impose the restriction $X_1 > 0$. Now introduce

$$Y = \alpha^2 (X + X_1)^{-1} > 0$$

to transform (4.22) into the design ARE (4.5). A solution $Y > 0$ of (4.5), with $\alpha^2 Y^{-1} \geq X$, guarantees $X_e \geq 0$ solves (4.18) when gains K, K_d, and L are computed from (4.19), (4.20), and (4.21). Hence, by Lemma 4.1 the closed-loop transfer-function matrix $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$, provided $(\tilde{F}_e, \tilde{H}_e)$ is a detectable pair.

The needed detectability condition is satisfied if (A, H) is a detectable pair and $A_\alpha = A + \alpha^{-2}GG^TX - BB^TX$ is Hurwitz. To see this, let $v^T = (v_1^T \;\; v_2^T)$ satisfy

$$\tilde{F}_e v = \begin{pmatrix} A - BB^TX & -BB^TX \\ \alpha^{-2}GG^TX & A + \alpha^{-2}GG^TX - LC \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda v, \tag{4.24}$$

$$\tilde{H}_e v = \begin{pmatrix} H & 0 \\ -B^TX & -B^TX \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0, \tag{4.25}$$

and try to show that $Re(\lambda) < 0$. The upper half of (4.24) and the lower half of (4.25) give $Av_1 = Hv_1$, while the upper part of (4.25) gives $Hv_1 = 0$. Since (A, H) is assumed a detectable pair, this implies either $Re(\lambda) < 0$ or $v_1 = 0$. Suppose $v_1 = 0$; then the lower half of (4.24) gives

$$\left(A + \frac{1}{\alpha^2}GG^TX - LC\right) v_2 = \lambda v_2. \tag{4.26}$$

Therefore, pre-multiplying (4.21) by v_2^* and post-multiplying by v_2, and using (4.23), gives

$$2Re(\lambda)v_2^*X_1v_2 + \frac{1}{\alpha^2}v_2^*X_1GG^TX_1v_2 + \alpha^2 v_2^*C^TCv_2 + v_2^*XBB^TXv_2 = 0. \tag{4.27}$$

Since every term but the first in (4.27) is nonnegative, the first term gives

$$Re(\lambda)v_2^*X_1v_2 \leq 0. \tag{4.28}$$

If inequality holds in (4.28), then $Re(\lambda) < 0$. If equality holds in (4.28), then every term in (4.27) is zero. Hence, $Cv_2 = 0$ and $B^TXv_2 = 0$, and thus (4.26) gives

$$\left(A + \alpha^{-2}GG^TX - BB^TX\right) v_2 = \lambda v_2.$$

By assumption, $A + \alpha^{-2}GG^TX - BB^TX$ is Hurwitz; therefore, $Re(\lambda) < 0$.

The following theorem summarizes the result.

Theorem 4.3. *Suppose (A, H) is a detectable pair, $X \geq 0$ satisfies the state-feedback design ARE (4.3) with $A_\alpha = A + \alpha^{-2}GG^TX - BB^TX$ Hurwitz, and $Y > 0$ satisfies the observer design ARE (4.5) with $\sigma_{\max}\{YX\} < \alpha^2$. If the observer gain is given by (4.8), then the dynamic controller (4.7) stabilizes the plant (4.1), and the closed-loop transfer-function matrix $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$.*

4.5 The Decentralized Control Design

The same approach applied to the centralized control problem in Section 4.4 is now applied to the decentralized problem. The design derived here also appears in [14].

Consider again the plant (4.1) with (A, H) a detectable pair. To allow the formulation of a q-channel decentralized control problem for the plant (4.1), adopt the following notation:

$$\sum_{i=1}^{q} B_i u_i = (B_1 \; B_2 \ldots B_q) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{pmatrix} \equiv Bu, \tag{4.29a}$$

$$y \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{pmatrix} x + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \end{pmatrix} \equiv Cx + w, \tag{4.29b}$$

$$w_e \equiv \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_q \end{pmatrix} = \begin{pmatrix} w_0 \\ w \end{pmatrix}, \tag{4.29c}$$

$$S_i = B_i B_i^T, \quad i \in \{1, 2, \ldots, q\}, \tag{4.29d}$$

$$S = S_1 + S_2 + \ldots + S_q = BB^T. \tag{4.29e}$$

The problem is to design a controller for each of the q control channels, where the i^{th} controller uses the local measurement y_i to generate the local control u_i for the plant.

The basic decentralized control law to be developed stabilizes the plant and provides a predetermined H_∞-norm bound for the closed-loop transfer-function matrix from w_e to z. The controllers which make up the control law are based on observers which form estimates ξ_i, $i \in \{1, 2, \ldots, q\}$, of the state x for feedback. The state estimates are used for feedback so as to approximate the state-feedback control

$$u = -B^T X x, \tag{4.30}$$

where $X \geq 0$ satisfies the ARE

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XSX + H^T H = 0. \tag{4.31}$$

That is, the i^{th} control is given by

$$u_i = -B_i^T X \xi_i, \tag{4.32}$$

which approximates a subvector of the state-feedback control (4.30). To mimic the plant dynamics, the i^{th} observer should ideally have the form

$$\dot{\xi}_i = A\xi_i + \sum_{j=1}^{q} B_j u_j + G w_0 + L_i(y_i - C_i \xi_i), \tag{4.33}$$

where L_i is some observer gain matrix. However, since the disturbance w_0 and the controls u_j, $j \neq i$, are not available to the observer, (4.33) cannot be implemented directly. Just as the centralized observer (4.16) uses (4.20) as an estimate of the worst disturbance, the i^{th} decentralized observer replaces w_0 in (4.33) by

$$\hat{w}_0^i = \frac{1}{\alpha^2} G^T X \xi_i. \tag{4.34}$$

The i^{th} observer also replaces u_j, $j \neq i$, by

$$\hat{u}_j^i = -B_j^T X \xi_i, \tag{4.35}$$

which are approximations, based on the state estimate of the i^{th} controller, of the controls applied by the other controllers according to their shared strategy. With the control (4.32), the observer structure (4.33), and estimates (4.34) and (4.35), the i^{th} controller becomes

$$\dot{\xi}_i = \left(A + \frac{1}{\alpha^2} G G^T X - SX - L_i C_i\right) \xi_i + L_i y_i \tag{4.36a}$$

$$u_i = -B_i^T X \xi_i, \tag{4.36b}$$

where the observer gains L_i, $i \in \{1, 2, \ldots, q\}$, are to be determined.

Applying the q controllers (4.36) to the plant (4.1) gives a closed-loop system of order $(q+1)n$ described by

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & -BB_c^T X_c \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & L_c \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} \equiv F_e x_e + G_e w_e \tag{4.37a}$$

$$z = \begin{pmatrix} H & 0 \\ 0 & -B_c^T X_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \equiv H_e x_e, \tag{4.37b}$$

where $\xi^T = (\xi_1^T \ \xi_2^T \ \dots \ \xi_q^T)$, and

$$A_{\alpha c} = \text{Diag}\,(A_\alpha, A_\alpha, \dots, A_\alpha) \tag{4.38a}$$

$$A_\alpha = A + \frac{1}{\alpha^2} GG^T X - SX, \tag{4.38b}$$

$$B_c = \text{Diag}\,(B_1, B_2, \dots, B_q), \tag{4.38c}$$

$$C_c = \text{Diag}\,(C_1, C_2, \dots, C_q), \tag{4.38d}$$

$$L_c = \text{Diag}\,(L_1, L_2, \dots, L_q), \tag{4.38e}$$

$$X_c = \text{Diag}\,(X, X, \dots, X). \tag{4.38f}$$

For convenience, define also

$$I_c^T = [I \ I \ \dots \ I] \in I\!R^{n \times qn}, \tag{4.38g}$$

$$G_c = I_c G, \tag{4.38h}$$

$$A_c = A_{\alpha c} + I_c B B_c^T X_c. \tag{4.38i}$$

Then, transforming coordinates of (4.37) such that the last qn state variables are the errors $e_i = \xi_i - x, \quad i \in \{1, 2, \dots, q\}$, gives

$$\dot{\tilde{x}}_e = \tilde{F}_e \tilde{x}_e + \tilde{G}_e w_e, \quad z = \tilde{H}_e \tilde{x}_e,$$

where

$$\tilde{F}_e = M_e^{-1} F_e M_e = \begin{pmatrix} A - SX & -BB_c^T X_c \\ \alpha^{-2} G_c G^T X & A_c - L_c C_c \end{pmatrix}, \tilde{G}_e = M_e^{-1} G_e = \begin{pmatrix} G & 0 \\ -G_c & L_c \end{pmatrix}, \tag{4.39a}$$

$$\tilde{H}_e = H_e M_e = \begin{pmatrix} H & 0 \\ -B^T X & -B_c^T X_c \end{pmatrix}, M_e = \begin{pmatrix} I & 0 \\ I_c & I \end{pmatrix}. \tag{4.39b}$$

The existence of a $(q+1)n \times (q+1)n$ matrix $X_e \geq 0$ satisfying

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = 0 \tag{4.40}$$

will by Lemma 4.1 guarantee stability and an H_∞-norm bound for the closed-loop system (4.37). Assume the form

$$\tilde{X}_e = \begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix}, \tag{4.41}$$

with $X \geq 0$ solving (4.31) and $X_1 > 0$ undetermined, and decompose the left-hand side of (4.40) into appropriately sized blocks as

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{pmatrix}. \tag{4.42}$$

Then, it turns out that the off-diagonal block U_{12} is identically zero, and that (4.31) gives $U_{11} = 0$. Hence, independent of L_c and X_1, (4.42) becomes

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = \begin{pmatrix} 0 & 0 \\ 0 & U_{22} \end{pmatrix},$$

with

$$U_{22} = (A_c - L_c C_c)^T X_1 + X_1 (A_c - L_c C_c) + \frac{1}{\alpha^2} X_1 (G_c G_c^T + L_c L_c^T) X_1 + X_c B_c B_c^T X_c.$$

Defining $W = \alpha^2 X_1^{-1}$, this reduces to

$$\begin{aligned} U_{22} &= \frac{1}{\alpha^2} X_1 \{ W A_c^T + A_c W + \frac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W \\ &+ G_c G_c^T + (L_c - W C_c^T)(L_c^T - C_c W) \} X_1. \end{aligned} \tag{4.43}$$

It is now possible to pick X_1 (or, equivalently, W) and L_c such that $U_{22} = 0$. While it is logical in view of Lemma 4.1 to try to eliminate the last term in (4.43), this is not generally possible, since L_c must be block-diagonal. Thus, L_c is chosen to eliminate the $n \times n$ main-diagonal blocks of $L_c - W C_c^T$. This requires

$$L_c = W_D C_c^T, \tag{4.44}$$

where W_D is given by

$$W = \begin{pmatrix} W_{11} & W_{12} & \dots & W_{1q} \\ W_{21} & W_{22} & \dots & W_{2q} \\ \vdots & \vdots & & \vdots \\ W_{q1} & W_{q2} & \dots & W_{qq} \end{pmatrix}, \quad W_D = \text{Diag}\,(W_{11}, W_{22}, \dots, W_{qq}),$$

or

$$L_i = W_{ii}C_i^T, \quad i \in \{1, 2, \ldots, q\}. \tag{4.45}$$

Then, (4.43) becomes

$$\begin{aligned} U_{22} &= \frac{1}{\alpha^2}X_1\{WA_c^T + A_cW + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW + G_cG_c^T \\ &+ (W - W_D)C_c^TC_c(W - W_D)\}X_1. \end{aligned} \tag{4.46}$$

Therefore, if $W > 0$ satisfies the Riccati-like algebraic equation

$$\begin{gathered} WA_c^T + A_cW + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW \\ + G_cG_c^T + (W - W_D)C_c^TC_c(W - W_D) = 0, \end{gathered} \tag{4.47}$$

then $U_{22} = 0$, and (4.40) is satisfied. Since $W > 0$ is required, $\tilde{X}_e \geq 0$ holds automatically, and by Lemma 4.1, $\tilde{F}_e$ is Hurwitz and $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$, provided $(\tilde{F}_e, \tilde{H}_e)$ is a detectable pair. The following lemma provides the needed result.

Lemma 4.2. *Given the definitions (4.38) and (4.39), where $X \geq 0$ satisfies (4.31), $W > 0$ satisfies (4.47), and L_c satisfies (4.44), the pair $(\tilde{F}_e, \tilde{H}_e)$ is detectable under the following three conditions:*

(i) (A, H) is a detectable pair;

(ii) $A_\alpha \equiv A + \alpha^{-2}GG^TX - SX$ is Hurwitz;

(iii) $A_\alpha + SX$ has no eigenvalues on the $j\omega$-axis.

Proof. Suppose λ is an eigenvalue of $\tilde{F}_e$ corresponding to an unobservable mode of $(\tilde{F}_e, \tilde{H}_e)$; that is, some $v^T = (v_1^T \ v_2^T) \neq 0$ satisfies

$$\tilde{F}_e v = \begin{pmatrix} A - BB^TX & -BB_c^TX_c \\ \alpha^{-2}G_cG^TX & A_c - L_cC_c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda v \tag{4.48}$$

and

$$\tilde{H}_e v = \begin{pmatrix} H & 0 \\ -B^TX & -B_c^TX_c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \tag{4.49}$$

The proof now consists of showing that $Re(\lambda) < 0$.

The lower block of (4.49) and the upper block of (4.48) combine to give $Av_1 = \lambda v_1$, while the upper block of (4.49) gives $Hv_1 = 0$. Since (A, H) is assumed a detectable pair, this implies that either $Re(\lambda) < 0$ or $v_1 = 0$. If $v_1 = 0$, then the lower block of (4.48) gives

$$(A_c - L_cC_c)v_2 = \lambda v_2. \tag{4.50}$$

The detectability proof is completed by showing that $A_c - L_cC_c$ is Hurwitz. The bracketed expression in (4.43) is equal to zero; therefore

$$(A_c - L_cC_c)W + W(A_c - L_cC_c)^T + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW + G_cG_c^T + L_cL_c^T = 0. \tag{4.51}$$

Let η^* be a left-eigenvector of $A_c - L_cC_c$ corresponding to the eigenvalue λ. Multiply (4.51) on the left by η^* and on the right by η to obtain

$$2Re(\lambda)\eta^*W\eta + \frac{1}{\alpha^2}\eta^*WX_cB_cB_c^TX_cW\eta + \eta^*G_cG_c^T\eta + \eta^*L_cL_c^T\eta = 0. \tag{4.52}$$

Since every other term in (4.52) is nonnegative, $Re(\lambda)\eta^*W\eta \leq 0$, with $W > 0$ assumed; therefore, $Re(\lambda) \leq 0$. The following argument demonstrates that $Re(\lambda) \neq 0$. If $Re(\lambda) = 0$, then every term in (4.52) must be zero; hence, $\eta^*L_c = 0$. Then λ is an eigenvalue of A_c. But a similarity transformation on A_c reveals that it can have no imaginary eigenvalues: If

$$M = \begin{pmatrix} I & & & \\ -I & I & & \\ \vdots & \vdots & \ddots & \\ -I & 0 & \dots & I \end{pmatrix},$$

then

$$M^{-1}A_cM = \begin{pmatrix} A_\alpha + SX & S_2X & \dots & S_qX \\ & A_\alpha & \dots & 0 \\ & & \ddots & \vdots \\ & & & A_\alpha \end{pmatrix},$$

where A_α is assumed Hurwitz, and $A_\alpha + SX$ is assumed to have no imaginary eigenvalues.

□

Under the conditions of Lemma 4.2, $\tilde{F}_e$ is Hurwitz by Lemma 4.1. Therefore, F_e is also Hurwitz, and the closed-loop transfer-function matrix $T(s) = H_e(sI - F_e)^{-1}G_e = \tilde{H}_e(sI -$

$\tilde{F}_e)^{-1}\tilde{G}_e$ from w_e to z satisfies $\|T\|_\infty \leq \alpha$. Condition (iii) of Lemma 4.2 is a new technical condition which must be introduced for the decentralized control problem.

The following theorem summarizes the result:

Theorem 4.4. *Let (A, H) be a detectable pair and α be a positive scalar. Suppose $X \geq 0$ satisfies*

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XSX + H^T H = 0, \tag{4.31}$$

$A_\alpha \equiv A + \alpha^{-2}GG^T X - SX$ is Hurwitz, and $A_\alpha + SX$ has no $j\omega$-axis eigenvalues. Let $W > 0$ satisfy the Riccati-like algebraic equation

$$\begin{aligned} &WA_c^T + A_c W + \frac{1}{\alpha^2} WX_c B_c B_c^T X_c W - WC_c^T C_c W \\ &+ G_c G_c^T + (W - W_D) C_c^T C_c (W - W_D) = 0. \end{aligned} \tag{4.47}$$

If the observer gains L_i, $i \in \{1, 2, \ldots, q\}$, are given by

$$L_i = W_{ii} C_i^T, \tag{4.45}$$

then the decentralized feedback control law

$$\dot{\xi}_i = \left(A + \frac{1}{\alpha^2} GG^T X - SX - L_i C_i\right)\xi_i + L_i y_i, \quad i \in \{1, 2, \ldots, q\}, \tag{4.36a}$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\}, \tag{4.36b}$$

stabilizes the plant (4.1), with decentralized structure given by (4.29), and the closed-loop transfer-function matrix

$$T(s) = H_e(sI - F_e)^{-1} G_e$$

from w_e to z (with F_e, G_e, and H_e defined in (4.37)) satisfies

$$\|T\|_\infty \leq \alpha.$$

4.6 Example 1

Consider the plant (4.1) with $q = 2$ and

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C_1 = [1\ 0\ 0\ 0] \quad C_2 = [0\ 0\ 1\ 0] \quad H = [1\ 0\ -1\ 0].$$

The spectrum of A is $\{-2.56, -1.32 \pm j2.92, +0.19\}$; hence, the plant has an unstable mode.

To compute a decentralized control for this plant, first form the coefficients of (4.47) from the plant matrices and the state-feedback design equation solution. Then, solve (4.47) by an iterative method: Compute an approximate solution W_0 by ignoring the complicating term $Q = (W - W_D)C_c^T C_c(W - W_D)$. Then use W_0 to compute an approximation of Q_0 of Q, and use Q_0 in the obvious way to compute the next approximate solution W_1. Iterate this procedure until the candidate solution W_i makes the matrix norm of left-hand side of (4.47) less than some acceptable tolerance; then take W_i as the solution W of (4.47). The tolerance used for this example was 0.001.

Table 4.1 compares the closed-loop eigenvalues and H_∞ norms of state-feedback designs with those of decentralized observer-based control designs for several values of α. For $\alpha > 4$, the state-feedback eigenvalues are easily recognizable in the spectra of the decentralized-control systems; for smaller α, more interaction with other poles is evident. The sequence of candidate solutions of the Riccati-like equation converges for $\alpha \geq 2$, while the state-feedback design Riccati equation has an appropriate solution for $\alpha \geq 1.3$.

4.7 Example 2

Consider the 5^{th}-order plant (4.1) with $q = 2$ and

$$A = \begin{pmatrix} 0 & 1 & 4 & -4 & 1 \\ -3 & -1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & -2 & -2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The spectrum of A is

$$\Lambda(A) = \{-0.0108 \pm j3.717, -3.7138, -1.5906, +1.3262\};$$

Table 4.1: Closed-loop spectra and H_∞ norms for varying α.

	State Feedback		Decentralized Control	
	Spectrum	$\|T\|_\infty$	Spectrum	$\|T\|_\infty$
$\alpha = 20$	-0.24 -2.54 -1.45$\pm j$2.98	2.30	-0.24 -2.52 -1.26$\pm j$2.90 -0.38 -2.54 -1.47$\pm j$2.97 -1.07 -2.70 -1.45$\pm j$2.98	3.64
$\alpha = 16$	-0.24 -2.54 -1.45$\pm j$2.98	2.30	-0.24 -2.52 -1.26$\pm j$2.90 -0.38 -2.54 -1.47$\pm j$2.97 -1.08 -2.70 -1.45$\pm j$2.98	3.63
$\alpha = 12$	-0.24 -2.54 -1.45$\pm j$2.98	2.29	-0.25 -2.52 -1.26$\pm j$2.90 -0.38 -2.54 -1.47$\pm j$2.97 -1.08 -2.70 -1.45$\pm j$2.98	3.59
$\alpha = 8$	-0.24 -2.54 -1.45$\pm j$2.98	2.27	-0.27 -2.52 -1.26$\pm j$2.90 -0.37 -2.54 -1.47$\pm j$2.97 -1.09 -2.70 -1.45$\pm j$2.98	3.49
$\alpha = 4$	-0.27 -2.54 -1.46$\pm j$2.98	2.15	-0.35$\pm j$0.08 -1.26$\pm j$2.91 -1.18 -2.54 -1.47$\pm j$2.97 -2.49 -2.71 -1.45$\pm j$2.98	3.05
$\alpha = 2$	-0.46 -2.54 -1.46$\pm j$2.98	1.76	-2.36$\pm j$0.85 -1.21$\pm j$2.98 -0.48 -2.53 -1.47$\pm j$2.98 -1.38 -2.79 -1.45$\pm j$2.94	1.995
$\alpha = 1.3$	-2.59 -3.11 -1.45$\pm j$2.94	1.30	none	none

hence, the plant has an unstable mode and a lightly-damped stable mode. This section gives the results of H_∞-suboptimal control designs for this plant. First, state-feedback solutions are presented, then observer-based solutions, both centralized and decentralized. For various values of the design parameter α, the spectrum, feedback and observer gains, and H_∞ norm for the closed-loop system are given.

4.7.1 State feedback

State-feedback designs can be computed for values of α varying from ∞ to 1.069199. For α = 1.069198, the solution X of the state-feedback design ARE (4.3) has a negative eigenvalue; hence, for all practical purposes, $\alpha_\infty = 1.069199$.

The closed-loop poles are the eigenvalues of $F = A - SX$. Figure 4.1 shows the position

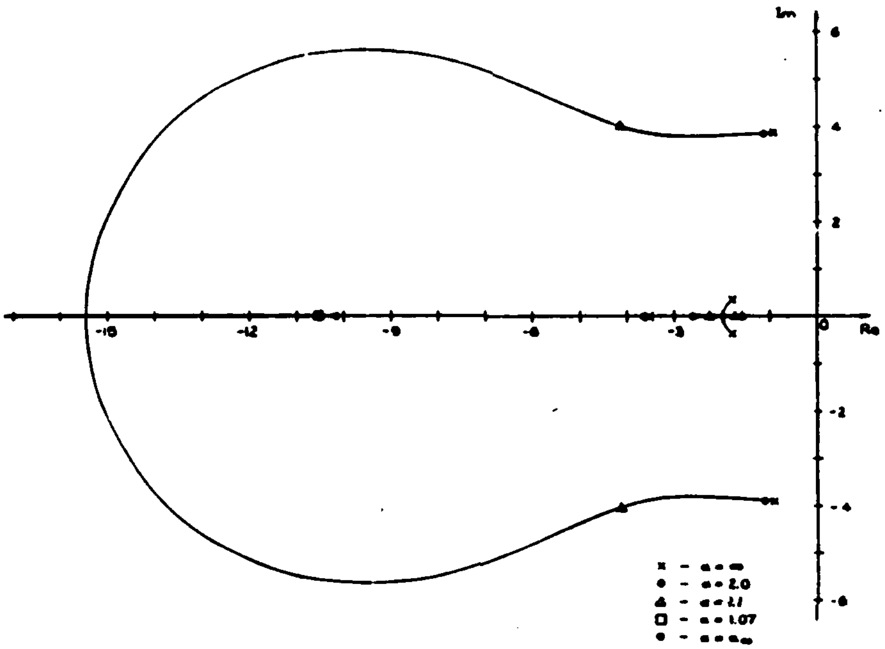

Figure 4.1: State-feedback poles for varying α Example 2.

of the closed-loop poles for a varying from ∞ to α_∞. Note that as α decreases from ∞ to 2.0, the poles barely move. As α decreases from 2.0 to 1.1, the most oscillatory mode is damped somewhat, and the other complex pole-pair meets at the real axis and splits into a real pair. Finally, as α decreases in the short interval from 1.1 to α_∞, the closed-loop poles are extremely sensitive to variations in α: The two remaining complex poles move leftward in the complex plane and meet at the real axis, then one pole goes toward $-\infty$. Naturally, moving a pole far into the left-half plane requires high feedback gains: The LQ feedback matrix is

$$K_{LQ} = \begin{pmatrix} -0.51 & -1.00 & -0.21 & -0.98 & -0.80 \\ -0.48 & -0.40 & 0.44 & -0.94 & -0.47 \end{pmatrix}$$

with resulting closed-loop spectrum

$$\Lambda(F) = \{-0.92 \pm j3.98, -1.78 \pm j0.35, -3.54\}$$

while a nearly H_∞-optimal ($\alpha = 1.07$) feedback matrix is

$$K = \begin{pmatrix} 13.21 & -3.67 & -51.59 & 61.76 & 7.93 \\ -66.51 & 3.97 & 189.88 & -235.86 & -40.62 \end{pmatrix}$$

with resulting spectrum

$$\Lambda(F) = \{-81.54, -10.51, -3.65, -2.63, -1.57\}.$$

These gains are much larger than the LQ gains, and they also have different signs. Reducing α to $\alpha_\infty = 1.069199$ results in gains (and one closed-loop pole) of magnitude larger than 10^5.

4.7.2 Centralized observer feedback

Observer-based centralized controls can be computed by the method given in Theorem 4.1 for values of α ranging from ∞ to 1.913. For $\alpha = 1.912$, the solutions X and Y of (4.3) and (4.5) do not satisfy the condition $\sigma_{\max}\{YX\} < \alpha^2$.

Figure 4.2 shows the position of the closed-loop poles for α varying from ∞ to 1.913. As α falls from ∞ to 3.0, the most oscillatory modes are damped somewhat, and all but the leftmost of the real poles move to the left on the real axis. As α falls from 3.0 to 2.4, the two leftmost real poles meet, split into a complex pair, circle leftward, meet again on the real axis, and move apart. Again, as α approaches its minimum, one pole moves off toward $-\infty$. As α decreases from ∞ to 1.913, each real-axis pole effectively shifts from its original LQG position to the LQG position vacated by the pole to its left, leaving the rightmost LQG position vacant and moving the leftmost real-axis pole toward $-\infty$.

The LQG ($\alpha = \infty$) observer-gain matrix is

$$L_{LQG} = \begin{pmatrix} 1.37 & -0.71 & 0.09 & 0.21 \\ -0.71 & 1.95 & 0.79 & 0.70 \\ 0.40 & 0.24 & 0.26 & 0.31 \\ 0.09 & 0.79 & 1.03 & 0.15 \\ 0.21 & 0.70 & 0.15 & 0.61 \end{pmatrix}$$

with resulting closed-loop spectrum

$$\Lambda(F_e) = \{-0.92 \pm j3.98, -1.78 \pm j0.35, -3.54, -1.09 \pm j3.82, -1.32, -1.69, -3.78\},$$

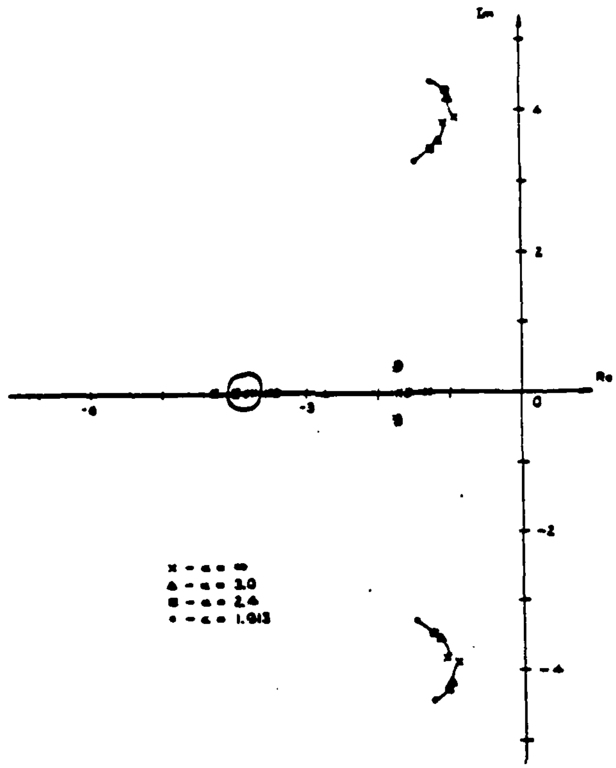

Figure 4.2: Output-feedback poles for a varying α, Example 2.

while the observer-gain matrix for $\alpha = 1.92$ is

$$L = \begin{pmatrix} 2.47 & 1.90 & 3.26 & 1.29 \\ 1.90 & 94.47 & 98.89 & 24.57 \\ 1.55 & 25.95 & 27.67 & 7.12 \\ 3.26 & 98.89 & 105.59 & 25.42 \\ 1.29 & 24.57 & 25.42 & 7.04 \end{pmatrix}$$

with resulting spectrum

$$\Lambda(F_e) = \{-204.31, -1.22 \pm j4.41, -1.47 \pm j3.29, -1.75 \pm j0.42, -3.83, -3.51, -1.65\}.$$

Reducing α to 1.913 results in some gains (and one pole) with magnitudes on the order of 10^3.

4.7.3 Decentralized control

Decentralized controls can be computed by the method given in Theorem 4.2 for values of α ranging from ∞ to 2.3323. The solution of the Riccati-like algebraic equation (4.47) is obtained using the simple iterative method described in Section 4.6. The smaller the value of α, the more iterations are required to obtain convergence: For example, to satisfy a tolerance of 0.001 on the largest singular value of the left-hand side of the Riccati-like equation, $\alpha = 10$ requires only 6 iterations, while $\alpha = 2.35$ requires 47 iterations. To speed up computations for small α, the solution for a slightly larger α can be used as the starting point; however, this "embedding" practice seems to result in convergence of the algorithm only when using the starting point $W = 0$ also results in convergence. For $\alpha = 2.3322$ and below, the algorithm does not seem to converge.

Figure 4.3 shows the position of the closed-loop poles for α varying from ∞ to 2.3323.

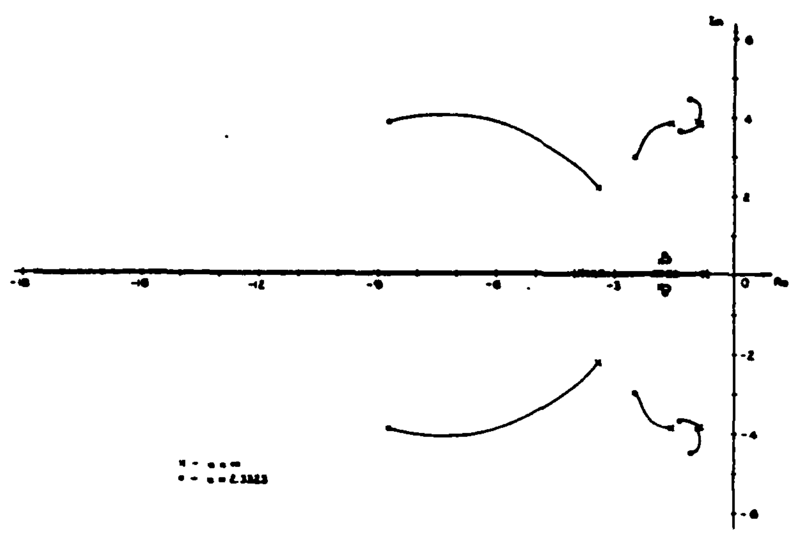

Figure 4.3: Closed-loop poles for decentralized control, Example 2.

As α decreases, the oscillatory modes are damped, and the real poles move to the left on the real axis. Again, as α approaches its minimum, the poles on the real axis seem to be shifting left into the positions originally occupied by other poles for $\alpha = \infty$.

For $\alpha = \infty$, the observer-gain matrices are

$$L_1 = \begin{pmatrix} 1.63 & -0.90 \\ -0.90 & 2.61 \\ 0.41 & 0.40 \\ -0.04 & 1.33 \\ 0.32 & 0.65 \end{pmatrix}, L_2 = \begin{pmatrix} 0.07 & -0.31 \\ 0.98 & 1.15 \\ 0.31 & 0.44 \\ 1.38 & 0.16 \\ 0.16 & 1.22 \end{pmatrix},$$

while the observer-gain matrices for $\alpha = 2.3323$ are

$$L_1 = \begin{pmatrix} 3.03 & -3.17 \\ -3.17 & 19.26 \\ 0.33 & 3.71 \\ -2.16 & 18.89 \\ 0.76 & 2.92 \end{pmatrix}, L_2 = \begin{pmatrix} 5.45 & 0.97 \\ 10.08 & 6.02 \\ 3.58 & 1.86 \\ 11.80 & 2.40 \\ 2.40 & 3.48 \end{pmatrix}.$$

Since the solution for $\alpha = 2.3323$ displays somewhat higher gains and an eigenvalue moving to the left, it seems a reasonable hypothesis that solutions may exist for smaller α, giving a high-gain result as in the state-feedback and centralized observer cases.

4.7.4 Spectrum and H_∞ norm comparisons

The spectra for state-feedback solutions and subspectra for centralized and decentralized observer-based solutions are shown for various values of α in Table 4.2. The state-feedback poles are recognizable among the poles of both observer-based solutions. Although the state-feedback root-locus plot (Fig. 4.1) appears quite different from the other two (Figs. 4.2 and 4.3), the observer-based solutions no longer exist when α is small enough that the state-feedback poles have moved significantly from their LQ positions.

The H_∞ norms of the closed-loop systems are compared for $\alpha \leq 5$ in Figure 4.4. The norms are seen to be monotone increasing with α. For $\alpha = \infty$, the H_∞ norms are $\|T\|_\infty = 1.55$ for state feedback, $\|T\|_\infty = 3.322$ for centralized observer feedback, and $\|T\|_\infty = 4.61$ for decentralized observer feedback, where $T(s)$ is the closed-loop transfer function matrix in each case. As the theory guarantees, the H_∞ norms are always smaller than the design

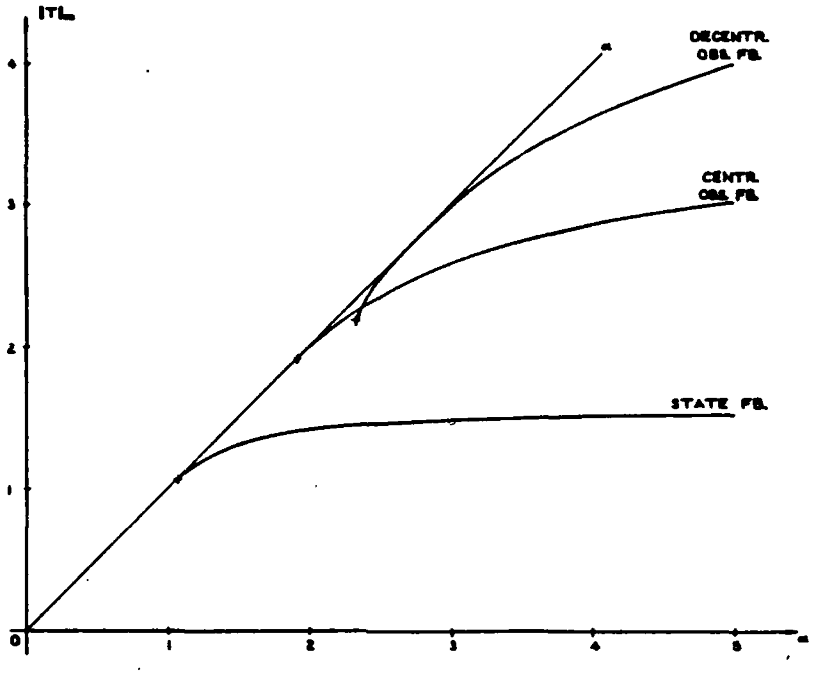

Figure 4.4: Comparison of actual closed-loop H_∞ norms, Example 2.

parameter α. In the state-feedback and centralized observer-based designs, the actual H_∞ norms and the bound α are very close for α close to the minimum value. In the decentralized case, the actual norm approaches the bound α in the neighborhood of $\alpha = 2.5$, then falls away slightly from the bound as α approaches the minimum value for which solutions of the Riccati-like design equation were computed. The "slack" in the bound suggests that decentralized designs guaranteeing smaller norms may exist, possibly corresponding with solutions of the Riccati-like equation for smaller values of α. Such solutions would have to be obtained by methods different from those used in this example.

Table 4.2: Closed-loop eigenvalues.

	State Feedback	Centralized Output Feedback	Decentralized Control
$\alpha = \infty$	-0.92$\pm j$3.89 -1.78$\pm j$0.35 - 3.54	-0.92$\pm j$3.89 -1.78$\pm j$0.35 -3.54 ...	-0.92$\pm j$3.89 -1.78$\pm j$0.35 -3.54 ...
$\alpha = 10$	-0.92$\pm j$3.89 -1.78$\pm j$0.35 - 3.54	-0.95$\pm j$3.93 -1.77$\pm j$0.34 -3.54 ...	-0.88$\pm j$3.94 -1.77$\pm j$0.34 -3.54 ...
$\alpha = 5$	-0.94$\pm j$3.89 -1.78$\pm j$0.35 - 3.54	-0.97$\pm j$4.03 -1.71$\pm j$0.35 -3.55 ...	-0.87$\pm j$4.02 -1.74$\pm j$0.32 -3.56 ...
$\alpha = 3$	-0.99$\pm j$3.89 -1.78$\pm j$0.35 - 3.54	-1.01$\pm j$4.17 -1.73$\pm j$0.43 -3.60 ...	-0.86$\pm j$4.18 -1.66$\pm j$0.46 -3.58 ...
$\alpha = 2.5$	-1.03$\pm j$3.89 -1.78$\pm j$0.36 - 3.54	-1.04$\pm j$4.26 -1.74$\pm j$0.43 -3.46 ...	-0.90$\pm j$4.35 -1.72$\pm j$0.46 -3.67 ...
$\alpha = 2$	-1.12$\pm j$3.89 -1.78$\pm j$0.36 -3.54	-1.17$\pm j$4.39 -1.75$\pm j$0.42 -3.51 ...	

5 RELIABLE CONTROL DESIGN

This section develops centralized and decentralized control designs which guarantee stability and a predetermined H_∞-norm bound despite measurement or control failures. Such designs are referred to here as "reliable".

There have been various attempts in the past to develop methodologies for the design of reliable control systems, and these attempts have had differing reliability goals. Among the most prominent attempts are [39], [58], [59], [60], [61], [62] and [63]. The design methodology presented here differs from all previous attempts in that it is the first to produce controls that guarantee stability and a H_∞-norm bound for the base case when all sensors and actuators are operative as well as in case of outages of certain sensors or certain actuators.

Section 5.1 presents an example which establishes the need for a reliable decentralized design. Section 5.2 develops centralized reliable designs, which guarantee stability and an H_∞-norm bound despite possible outages of sensors or actuators within predefined susceptible sets. The cases of sensor and actuator outages are treated separately, resulting in two designs with different reliability properties. Section 5.3 presents decentralized reliable designs which guarantee stability and an H_∞-norm bound despite possible outages of certain control channels in the decentralized system. The control channel outages are modelled first as measurement outages, and then as control input outages, resulting in two distinct designs with the same reliability properties. Section 5.4 present results on the design of strongly stable systems.

5.1 Motivation

The 4^{th}-order example of Section 4.6 is used to motivate the development of a reliable decentralized control. In this example, stability and a predetermined H_∞-norm bound are guaranteed by the basic decentralized design for various values of the design parameter α. Table 5.1 gives the actual H_∞ norms of the closed-loop systems corresponding with several values of α. In addition to the case when no controller failure occurs, Table 5.1 gives the conditions corresponding with a failure of each of the two controllers. A failure of Controller

Table 5.1: H_∞ norms for the basic decentralized design.

	no failure	#1 fails	#2 fails
$\alpha = 20$	3.64	unstable	5.34
$\alpha = 16$	3.63	unstable	5.30
$\alpha = 14$	3.61	unstable	5.28
$\alpha = 12$	3.59	unstable	5.23
$\alpha = 8$	3.49	unstable	5.04
$\alpha = 4$	3.05	unstable	4.19
$\alpha = 2$	1.995	unstable	2.46

#1 results in instability for each design computed, while a failure of Controller #2 results only in an increased H_∞ norm for the closed-loop system.

Since the plant is open-loop unstable, a failure of both controllers at once necessarily results in instability; however, it would be desirable to alter the design so as to guarantee at least stability, and, better still, some level of disturbance attenuation for the closed-loop system if only one controller should fail. While the basic design in this case still works well if only Controller #2 fails, it is not acceptable if Controller #1 fails. Therefore, a design reliable with respect to failure in Controller #1 is desired.

The essential idea in developing a reliable design methodology is that, if there exists $X_e \geq 0$ satisfying

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = 0 \qquad (5.1)$$

with some $P_e \geq 0$, then the resulting closed-loop system will by Lemma 4.1 be stable and have H_∞-norm bound α. Choosing $P_e = 0$ in (5.1) yields the basic centralized and decentralized designs derived in Section 4, characterized by closed-loop stability and the bound $\|T\|_\infty \leq \alpha$. Identification of the appropriate P_e can ensure additional system properties associated with reliability. It turns out that the appropriate choices of P_e introduce perturbations into the basic design equations equivalent to appending columns or rows to G or H in the basic design equations. Preliminary results on design of reliable control systems were presented in [15].

5.2 Reliable Centralized Design

The problem addressed here is that of designing a centralized controller which is reliable despite possible sensor or actuator outages. The outages will be restricted to occur within a preselected subset of available measurements or control inputs. The controllers developed will guarantee closed-loop stability and a predetermined H_∞-norm bound, regardless of admissible sensor or actuator failures. The cases of sensor and actuator outages are treated separately, and two designs are developed to handle the two cases. However, it will be clear from the results that controllers which can handle both sensor and actuator outages can be obtained by combining the designs.

Consider first the design of a controller that can tolerate the outage of certain sensors which provide the various elements of the measurement vector y. Let $\Omega \subseteq \{1, 2, \ldots, \dim(y)\}$ correspond with a selected subset of sensors susceptible to outages. Introduce the decomposition

$$C = C_\Omega + C_{\bar{\Omega}}, \tag{5.2}$$

where C_Ω denotes the measurement matrix associated with Ω, and $C_{\bar{\Omega}}$ denotes the measurement matrix associated with the complementary subset of measurements. In other words, $C_{\bar{\Omega}}$ is the same as C, but with rows corresponding with susceptible sensors zeroed out. Let $\omega \subseteq \Omega$ correspond with a particular subset of the susceptible sensors that actually experience an outage, and let $T_{\bar{\omega}}(s)$ denote the transfer-function matrix of the resulting closed-loop system. It is convenient to adopt the notation

$$C = C_\omega + C_{\bar{\omega}} \tag{5.3}$$

where C_ω and $C_{\bar{\omega}}$ have meanings analogous to those of C_Ω and $C_{\bar{\Omega}}$ in (5.2). Since $\omega \subseteq \Omega$, $C_\omega^T C_\omega \leq C_\Omega^T C_\Omega$. Also decompose the observer gain as

$$L = L_\omega + L_{\bar{\omega}} \tag{5.4}$$

so that

$$LC = L_\omega C_\omega + L_{\bar{\omega}} C_{\bar{\omega}}.$$

($L_{\bar{\omega}}$ has columns zeroed out corresponding with sensors which have actually failed.) Then the following result holds:

Theorem 5.1. *With all assumptions and the design otherwise as in Theorem 4.3, assume $X \geq 0$ and $Y > 0$ satisfy the AREs*

$$A^T X + XA - XSX + \frac{1}{\alpha^2} XGG^T X + H^T H + \alpha^2 C_\Omega^T C_\Omega = 0, \tag{5.5}$$

$$AY + YA^T + \frac{1}{\alpha^2} YH^T HY - YC_\Omega^T C_\Omega Y + GG^T = 0, \tag{5.6}$$

respectively. Then, for sensor outages corresponding with any $\omega \subseteq \Omega$, the closed-loop system is stable, and $\|T_{\bar{\omega}}\|_\infty \leq \alpha$.

Remark 5.1: With all sensors operational, corresponding with $\omega = \emptyset$, $T_{\bar{\omega}}(s) = T(s)$ is the transfer-function matrix from w_e to z, where

$$w_e = \begin{pmatrix} w_0 \\ w \end{pmatrix}, \quad z = \begin{pmatrix} Hx \\ u \end{pmatrix}.$$

Theorem 5.1 covers this case automatically, since $\omega = \emptyset \subseteq \Omega$. If sensors corresponding to a nonempty subset $\omega \subseteq \Omega$ fail, then $T_{\bar{\omega}}(s)$ is the transfer-function matrix from $w_{e\bar{\omega}}$ to z, where

$$w_{e\bar{\omega}} = \begin{pmatrix} w_0 \\ w_{\bar{\omega}} \end{pmatrix},$$

with $w_{\bar{\omega}}$ containing only those components of measurement noise associated with operational sensors.

Proof: The design equations (5.5) and (5.6) arise from replacing H in the description of the plant by the augmented matrix

$$H_+ = \begin{pmatrix} H \\ \alpha C_\Omega \end{pmatrix}, \tag{5.7}$$

and changing the design equations accordingly. If (5.5) and (5.6) have appropriate solutions, then Theorem 4.3 guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e+}^T H_{e+} = 0, \tag{5.8}$$

where the augmented closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, H_{e+} = \begin{pmatrix} H_+ & 0 \\ 0 & -B^T X \end{pmatrix}, \tag{5.9}$$

and (F_e, H_{e+}) is a detectable pair. The actual closed-loop system with no sensor outages is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ 0 & -B^T X \end{pmatrix}. \tag{5.10}$$

For sensor outages corresponding with $\omega \subseteq \Omega$, the controller becomes

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2} G G^T X - SX - LC\right) \xi + L_{\bar{\omega}} y, \tag{5.11a}$$

$$u = -B^T X \xi. \tag{5.11b}$$

The controller dynamic structure is not affected by a sensor outage; only the controller input structure is effectively changed. Given (5.11), the closed-loop system matrices become

$$F_{e\bar{\omega}} = \begin{pmatrix} A & -SX \\ L_{\bar{\omega}} C_{\bar{\omega}} & A_\alpha - LC \end{pmatrix}, G_{e\bar{\omega}} = \begin{pmatrix} G & 0 \\ 0 & L_{\bar{\omega}} \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ 0 & -B^T X \end{pmatrix}. \tag{5.12}$$

The following useful relations are derived from (5.9), (5.10), and (5.12):

$$F_e = F_{e\bar{\omega}} + \begin{pmatrix} 0 \\ L_w \end{pmatrix} (C_\omega \ 0) \equiv F_{e\bar{\omega}} + L_{e\omega} C_{e\omega}, \tag{5.13a}$$

$$G_e G_e^T = \begin{pmatrix} G G^T & 0 \\ 0 & L_{\bar{\omega}} L_{\bar{\omega}}^T \end{pmatrix} + \begin{pmatrix} 0 \\ L_\omega \end{pmatrix} (0 \ L_\omega^T) = G_{e\bar{\omega}} G_{e\bar{\omega}}^T + L_{e\omega} L_{e\omega}^T, \tag{5.13b}$$

$$H_{e+}^T H_{e+} = H_e^T H_e + \alpha^2 \begin{pmatrix} C_\Omega^T C_\Omega & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.13c}$$

Use (5.8) and (5.13) to obtain

$$\begin{aligned} & F_{e\bar{\omega}}^T X_e + X_e F_{e\bar{\omega}} + \frac{1}{\alpha^2} X_e G_{e\bar{\omega}} G_{e\bar{\omega}}^T X_e + H_e^T H_e \\ & \quad = -C_{e\omega}^T L_{e\omega}^T X_e - X_e L_{e\omega} C_{e\omega} - \frac{1}{\alpha^2} X_e L_{e\omega} L_{e\omega}^T X_e - \alpha^2 \begin{pmatrix} C_\Omega^T \\ 0 \end{pmatrix} (C_\Omega \ 0). \end{aligned} \tag{5.14}$$

Therefore, since $-C_\Omega^T C_\Omega \leq -C_\omega^T C_\omega$, (5.14) gives

$$\begin{aligned} F_{e\bar{\omega}}^T X_e \quad & + \quad X_e F_{e\bar{\omega}} + \frac{1}{\alpha^2} X_e G_{e\bar{\omega}} G_{e\bar{\omega}}^T X_e + H_e^T H_e \\ & \leq \quad -C_{e\omega}^T L_{e\omega}^T X_e - X_e L_{e\omega} C_{e\omega} - \frac{1}{\alpha^2} X_e L_{e\omega} L_{e\omega}^T X_e - \alpha^2 C_{e\omega}^T C_{e\omega} \\ & = \quad -\left(\frac{1}{\alpha} X_e L_{e\omega} + \alpha C_{e\omega}^T\right) \left(\frac{1}{\alpha} L_{e\omega}^T X_e + \alpha C_{e\omega}\right) \leq 0. \end{aligned} \tag{5.15}$$

Hence, provided $(F_{e\bar{\omega}}, H_e)$ is a detectable pair, Lemma 4.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_e(sI - F_{e\bar{\omega}})^{-1}G_{e\bar{\omega}}$, the transfer-function matrix from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$, satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$. The detectability proof is routine: If $v^T = (v_1^T \; v_2^T) \neq 0$ satisfies $F_{e\bar{\omega}}v = \lambda v$ and $H_e v = 0$, then $Av_1 = \lambda v_1$ and $Hv_1 = 0$, with (A, H) assumed a detectable pair. Therefore, either $Re(\lambda) < 0$ or $v_1 = 0$. Suppose $v_1 = 0$; then $F_e v = F_{e\bar{\omega}}v = \lambda v$ and $H_e v = 0$ gives $H_{e+}v = 0$. Since (F_e, H_{e+}) is a detectable pair, $Re(\lambda) < 0$. □

Consider now the design of a controller that can tolerate the outage of certain actuators which provide the various elements of the control vector u. Let $\Omega \subseteq \{1, 2, ..., \dim(u)\}$ correspond with a selected subset of actuators susceptible to outages. Introduce the decomposition

$$B = B_\Omega + B_{\bar{\Omega}}, \tag{5.16}$$

where B_Ω denotes the control matrix associated with the set Ω, and $B_{\bar{\Omega}}$ denotes the control matrix associated with the complementary subset of control inputs. In other words, $B_{\bar{\Omega}}$ is the same as B, but with columns corresponding with susceptible actuators zeroed out. Let $\omega \subseteq \Omega$ correspond with a particular subset of the susceptible actuators that actually fail, and let $T_{\bar{\omega}}(s)$ denote the transfer-function matrix of the resulting closed-loop system. It is convenient to adopt the notation

$$B = B_\omega + B_{\bar{\omega}} \tag{5.17}$$

where B_ω and $B_{\bar{\omega}}$ have meanings analogous to those of B_Ω and $B_{\bar{\Omega}}$ in (5.16). Since $\omega \subseteq \Omega$, $B_\omega B_\omega^T \leq B_\Omega B_\Omega^T$. Then the following result, dual to Theorem 5.1, holds:

Theorem 5.2. *With all assumptions and the design otherwise as in Theorem 4.3, assume* $X \geq 0$ *and* $Y > 0$ *satisfy the AREs*

$$A^T X + XA - XB_{\bar{\Omega}}B_{\bar{\Omega}}^T X + \frac{1}{\alpha^2}XGG^T X + H^T H = 0, \tag{5.18}$$

$$AY + YA^T + \frac{1}{\alpha^2}YH^T HY - YC^T CY + GG^T + \alpha^2 B_\Omega B_\Omega^T = 0, \tag{5.19}$$

respectively. Define

$$G_+ = (G \; \alpha B_\Omega), \tag{5.20}$$

and let the controller be given by

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2}G_+G_+^TX - SX - LC\right)\xi + Ly, \tag{5.21a}$$

$$u = -B^TX\xi. \tag{5.21b}$$

Assume the controller is open-loop (internally) stable. Then, for actuator outages corresponding with any $\omega \subseteq \Omega$, the closed-loop system is stable, and $\|T_{\bar{\omega}}\|_\infty \leq \alpha$.

Remark 5.2: For actuator outages corresponding with $\omega \subseteq \Omega$, $T_{\bar{\omega}}(s)$ is the transfer-function matrix from w_e to $z_{\bar{\omega}}$, where $z_{\bar{\omega}}$ excludes control components associated with failed actuators.

Proof: The design equations (5.18) and (5.19) arise from replacing the matrix G in the description of the plant (4.1) with the augmented matrix G_+, and introducing the corresponding changes in the design equations. If (5.18) and (5.19) have appropriate solutions, then Theorem 4.1 guarantees that $X_e \geq 0$ satisfies

$$F_e^TX_e + X_eF_e + \frac{1}{\alpha^2}X_eG_{e+}G_{e+}^TX_e + H_e^TH_e = 0, \tag{5.22}$$

where the augmented closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, G_{e+} = \begin{pmatrix} G_+ & 0 \\ 0 & L \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ 0 & -B^TX \end{pmatrix}, \tag{5.23}$$

with $A_\alpha \equiv A + \alpha^{-2}G_+G_+^TX - SX$ and (F_e, H_e) a detectable pair. When there are no actuator outages, the actual closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ 0 & -B^TX \end{pmatrix}. \tag{5.24}$$

For actuator outages corresponding with $\omega \subseteq \Omega$, the controller becomes

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2}G_+G_+^TX - SX - LC\right)\xi + Ly, \tag{5.25a}$$

$$u = -B_{\bar{\omega}}^TX\xi. \tag{5.25b}$$

The controller dynamic structure is not affected by actuator outages; only the controller output structure is effectively changed. Given (5.25), the closed-loop system is described by the matrices

$$F_{e\bar{\omega}} = \begin{pmatrix} A & -B_{\bar{\omega}}B_{\bar{\omega}}^TX \\ LC & A_\alpha - LC \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, H_{e\bar{\omega}} = \begin{pmatrix} H & 0 \\ 0 & -B_{\bar{\omega}}^TX \end{pmatrix}. \tag{5.26}$$

The following useful relations are derived from (5.23), (5.24), and (5.26):

$$F_e = F_{e\bar{\omega}} - \begin{pmatrix} B_\omega \\ 0 \end{pmatrix}(0\ B_\omega^T X) \equiv F_{e\bar{\omega}} - B_{e\omega}(0\ B_\omega^T X), \tag{5.27a}$$

$$H_e^T H_e = H_{e\bar{\omega}}^T H_{e\bar{\omega}} + \begin{pmatrix} 0 \\ XB_\omega \end{pmatrix}(0\ B_\omega^T X), \tag{5.27b}$$

$$G_{e+}G_{e+}^T = G_e G_e^T + \alpha^2 \begin{pmatrix} B_\Omega B_\Omega^T & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.27c}$$

Use (5.22) and (5.27) to obtain

$$\begin{aligned} F_{e\bar{\omega}}^T X_e \quad &+ \quad X_e F_{e\bar{\omega}} + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e\bar{\omega}}^T H_{e\bar{\omega}} \\ &\le \quad -\left(X_e B_{e\omega} - \left(\begin{smallmatrix} 0 \\ XB_\omega \end{smallmatrix}\right)\right)\left(B_{e\omega}^T X_e - \left(0\ B_\omega^T X\right)\right) \le 0. \end{aligned} \tag{5.28}$$

Provided $(F_{e\bar{\omega}}, H_{e\bar{\omega}})$ is a detectable pair, Lemma 4.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_{e\bar{\omega}}(sI - F_{e\bar{\omega}})^{-1}G_e$ satisfies $\|T_{\bar{\omega}}\|_\infty \le \alpha$. To prove detectability, let $v^T = (v_1^T\ v_2^T) \ne 0$ satisfy $F_{e\bar{\omega}}v = \lambda v$ and $H_{e\bar{\omega}}v = 0$; then $Av_1 = \lambda v_1$ and $Hv_1 = 0$, with (A, H) assumed a detectable pair. Therefore, either $Re(\lambda) < 0$ or $v_1 = 0$. If $v_1 = 0$, then $F_{e\bar{\omega}}v = \lambda v$ gives

$$\left(A + \frac{1}{\alpha^2}G_+G_+^T X - SX - LC\right)v_2 = \lambda v_2. \tag{5.29}$$

By the assumption that the controller is open-loop stable, $(A + \alpha^{-2}G_+G_+^T X - SX - LC)$ is Hurwitz; therefore, $Re(\lambda) < 0$. □

The design given in Theorem 5.2, unlike that given in Theorem 5.1, requires that the controller turn out stable in order to guarantee reliable closed-loop stability. If the design does not result in a stable controller, it may be combined with a strongly stabilizing design developed in Section 5.5; then the assumption of open-loop stability of the controller will hold automatically.

Note that to achieve reliability with respect to sensor outages, it is sufficient to modify the feedback and observer gains; however, to achieve reliability with respect to actuator outages, the observer structure must also be modified. The structural modification required is the inclusion of G_+ in the controller dynamic matrix.

5.3 Reliable Decentralized Design

Let $\Omega \subseteq \{1, 2, \dots, q\}$ correspond with a subset of controllers subject to outages. The problem is to compute a decentralized control law which guarantees closed-loop stability and an H_∞-norm bound in spite of controller outages corresponding with any subset $\omega \subseteq \Omega$. Without loss of generality, $\Omega = \{t+1, t+2, \dots, q\}$ and $\omega = \{r+1, r+2, \dots, q\}$, with $r \geq t$. Introduce the decompositions

$$B = (B_1 \dots B_r \;\; 0 \dots 0) + (0 \dots 0 \;\; B_{r+1} \dots B_q) \equiv B_{\bar{\omega}} + B_\omega, \tag{5.30a}$$

$$B_c = \mathrm{Diag}(B_1, \dots, B_r, 0, \dots, 0) + \mathrm{Diag}(0, \dots, 0, B_{r+1}, \dots, B_q) \equiv B_{c\bar{\omega}} + B_{c\omega}, \tag{5.30b}$$

$$C = \begin{pmatrix} C_1 \\ \vdots \\ C_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ C_{r+1} \\ \vdots \\ C_q \end{pmatrix} \equiv C_{\bar{\omega}} + C_\omega, \tag{5.30c}$$

$$C_c = \mathrm{Diag}(C_1, \dots, C_r, 0, \dots, 0) + \mathrm{Diag}(0, \dots, 0, C_{r+1}, \dots, C_q) \equiv C_{c\bar{\omega}} + C_{c\omega}, \tag{5.30d}$$

$$L_c = \mathrm{Diag}(L_1, \dots, L_r, 0, \dots, 0) + \mathrm{Diag}(0, \dots, 0, L_{r+1}, \dots, L_q) \equiv L_{c\bar{\omega}} + L_{c\omega}. \tag{5.30e}$$

Also decompose the disturbance and regulated output vectors as

$$w_e = \begin{pmatrix} w_0 \\ w \end{pmatrix} = \begin{pmatrix} w_0 \\ w_{\bar{\omega}} \\ w_\omega \end{pmatrix} = \begin{pmatrix} w_{e\bar{\omega}} \\ w_\omega \end{pmatrix}, \tag{5.31a}$$

$$z = \begin{pmatrix} Hx \\ u_{\bar{\omega}} \\ u_\omega \end{pmatrix} = \begin{pmatrix} z_{\bar{\omega}} \\ u_\omega \end{pmatrix}. \tag{5.31b}$$

Finally, define

$$B_\Omega = (B_{t+1} \dots B_q), \tag{5.32a}$$

$$C_\Omega^T = (C_{t+1}^T \dots C_q^T). \tag{5.32b}$$

Note that for any $\omega \subseteq \Omega$,

$$B_\Omega B_\Omega^T \geq B_\omega B_\omega^T, \tag{5.33}$$

$$C_\Omega^T C_\Omega \geq C_\omega^T C_\omega. \tag{5.34}$$

When no controller failures occur, the closed-loop system is described by matrices of the form

$$F_e = \begin{pmatrix} A & -BB_c^T \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ 0 & L_c \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ 0 & -B_c^T X_c \end{pmatrix},$$

where $A_{\alpha c} = \mathrm{Diag}(A_\alpha, A_\alpha, \ldots, A_\alpha)$. Suppose that controller failures take the form

$$y_i = 0, \quad i \in \omega. \tag{5.35}$$

The closed-loop system then takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & -BB^T X_c \\ L_{c\bar{\omega}} C_{\bar{\omega}} & A_{\alpha c} - L_c C_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & L_{c\bar{\omega}} \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} = F_{e\bar{\omega}} x_e + G_{e\bar{\omega}} w_e, \tag{5.36a}$$

$$z = \begin{pmatrix} H & 0 \\ 0 & -B_c^T X_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = H_e x_e. \tag{5.36b}$$

Because of the assumed mode of failure, given by (5.35), the disturbances w_i, $i \in \omega$, do not enter the system (5.36). In fact, (5.36) is a controllability canonical form, with ξ_i, $i \in \omega$, the uncontrollable parts of the extended state vector. Note also that

$$A_{\alpha c} - L_c C_c = \mathrm{Diag}(A_\alpha - L_1 C_1, A_\alpha - L_2 C_2, \ldots, A_\alpha - L_q C_q), \tag{5.37}$$

where $A_\alpha - L_i C_i$ is the open-loop dynamic matrix of the i^{th} controller. Because of the form of (5.36), the open-loop eigenvalues of the controllers which have failed appear directly as modes of the closed-loop system. This means that a design guaranteeing reliable stability will automatically guarantee that all controllers susceptible to outages are open-loop stable.

It is convenient to note that $F_{e\bar{\omega}}$ and $G_{e\bar{\omega}}$ are related to F_e and G_e by

$$F_{e\bar{\omega}} = F_e - \begin{pmatrix} 0 \\ L_{c\omega} \end{pmatrix} (C_\omega \;\; 0) \equiv F_e - L_{e\omega} C_{e\omega}, \tag{5.38a}$$

$$G_{e\bar{\omega}} = G_e - \begin{pmatrix} 0 & 0 \\ 0 & L_{c\omega} \end{pmatrix}, \tag{5.38b}$$

$$G_{e\bar{\omega}} G_{e\bar{\omega}}^T = G_e G_e^T - L_{e\omega} L_{e\omega}^T. \tag{5.38c}$$

The design which follows will guarantee that $F_{e\bar{\omega}}$ is Hurwitz, and that the transfer-function matrix $T_{\bar{\omega}}(s) = H_e(sI - F_{e\bar{\omega}})^{-1} G_{e\bar{\omega}}$ satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$, for controller outages associated with any $\omega \subseteq \Omega$. The case where no controllers fail (represented by $\omega = \emptyset \subseteq \Omega$) is always admissible; hence, the design will automatically guarantee that F_e is Hurwitz and that $T(s) = H_e(sI - F_e)^{-1}G_e$ satisfies $\|T\|_\infty \leq \alpha$. The following theorem gives the reliable design.

Theorem 5.3. *With all assumptions and the decentralized design otherwise as in Theorem 4.4, let $X \geq 0$ satisfy*

$$A^TX + XA + \frac{1}{\alpha^2}XGG^TX - XBB^TX + H^TH + \alpha^2 C_\Omega^T C_\Omega = 0, \tag{5.39}$$

where $\Omega \subseteq \{1, 2, \ldots, q\}$. Then, for controller outages corresponding with any $\omega \subseteq \Omega$, the closed-loop system (5.8) is internally stable, and the closed-loop transfer-function matrix $T_{\bar{\omega}}(s)$ from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$ satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$. In addition, all controllers corresponding with the "susceptible" set Ω are open-loop stable.

Remark 5.3: The design given in Theorem 5.3 results from replacing H in the description of the plant (4.1) with the augmented matrix

$$H_+ = \begin{pmatrix} H \\ \alpha C_\Omega \end{pmatrix}, \tag{5.40}$$

and changing the design equations accordingly. This substitution results in no change in the design equation (4.47), and is equivalent to selecting P_e in (5.1) as

$$P_e = \begin{pmatrix} \alpha^2 C_\Omega^T C_\Omega & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \tag{5.41}$$

The basic decentralized design computed for the augmented plant will provide reliable control for the actual plant.

Proof: Just as in the development of Section 4.5, the existence of appropriate solutions to the perturbed design equations (5.39) and (4.44) guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e+}^T H_{e+} = 0, \tag{5.42}$$

where

$$H_{e+} = \begin{pmatrix} H_+ & 0 \\ 0 & -B_c^T X_c \end{pmatrix}. \tag{5.43}$$

Now (5.38), (5.40), (5.42), and (5.43) give

$$\begin{aligned} F_{e\bar{\omega}}^T X_e &+ X_e F_{e\bar{\omega}} + \frac{1}{\alpha^2} X_e G_{e\bar{\omega}} G_{e\bar{\omega}}^T X_e + H_e^T H_e \\ &= -C_{e\omega}^T L_{e\omega}^T X_e - X_e L_{e\omega} C_{e\omega} - \frac{1}{\alpha^2} X_e L_{e\omega} L_{e\omega}^T X_e - \alpha^2 \begin{pmatrix} C_\Omega^T \\ 0 \end{pmatrix} (C_\Omega \ 0). \end{aligned}$$

Therefore, by (5.34),

$$\begin{aligned} F_{e\bar{\omega}}^T X_e &+ X_e F_{e\bar{\omega}} + \frac{1}{\alpha^2} X_e G_{e\bar{\omega}} G_{e\bar{\omega}}^T X_e + H_e^T H_e \\ &\leq -C_{e\omega}^T L_{e\omega}^T X_e - X_e L_{e\omega} C_{e\omega} - \frac{1}{\alpha^2} X_e L_{e\omega} L_{e\omega}^T X_e - \alpha^2 C_{e\omega}^T C_{e\omega} \\ &= -\left(\frac{1}{\alpha} X_e L_{e\omega} + \alpha C_{e\omega}^T\right) \left(\frac{1}{\alpha} L_{e\omega}^T X_e + \alpha C_{e\omega}\right) \leq 0. \end{aligned}$$

Hence, provided $(F_{e\bar{\omega}}, H_e)$ is a detectable pair, Lemma 4.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_e(Is - F_{e\bar{\omega}})^{-1} G_{e\bar{\omega}}$, the transfer-function matrix from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$, satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$. The detectability proof is the same as that of Lemma 4.2: Assuming $v \neq 0$ satisfies $F_{e\bar{\omega}} v = \lambda v$ and $H_e v = 0$ gives $A v_1 = \lambda v_1$ and $H v_1 = 0$, with (A, H) assumed a detectable pair. Therefore, either $Re(\lambda) < 0$ or $v_1 = 0$. If $v_1 = 0$, then $(A_{\alpha c} - L_c C_c) v_2 = \lambda v_2$, and hence $(A_c - L_c C_c) v_2 = \lambda v_2$, where $A_c - L_c C_c$ is known to be Hurwitz. □

Recall that the closed-loop system (5.36) assumes measurement failures corresponding with each $i \in \omega$. If instead there are control input failures, that is, if the controller failures are given by

$$u_i = 0, \quad i \in \omega, \tag{5.44}$$

then the closed-loop system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & -B_{\bar{\omega}} B_{c\bar{\omega}}^T X_c \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & L_c \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} \equiv F_{e\bar{\omega}} x_e + G_e w_e \tag{5.45a}$$

$$z = \begin{pmatrix} H & 0 \\ 0 & -B_{c\bar{\omega}}^T X_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \equiv H_{e\bar{\omega}} x_e, \tag{5.45b}$$

where $F_{e\bar{\omega}}$ has been redefined. Note that (5.45) is an observability canonical form, with ξ, $i \in \omega$, the unobservable parts of the extended state vector. In fact, for a given decentralized control law, (5.36) and (5.45) are just two different realizations of the same transfer-function matrix. However, the form (5.45) leads to the need for a different matrix P_e in (5.1) to guarantee reliable stability and performance, and hence to a different control law. Again, the closed-loop eigenvalues of the controllers which fail appear directly as modes of the closed-loop system; unlike the proof of Theorem 5.3, however, the following development must assume that all the controllers turn out open-loop stable. If some controllers turn out unstable, the design of Theorem 5.4 may be combined with a strongly stabilizing decentralized design developed in Section 5.5.

It is convenient to note that $F_{e\bar{\omega}}$ and $H_{e\bar{\omega}}$ are related to F_e and H_e by

$$F_{e\bar{\omega}} = F_e + \begin{pmatrix} B_\omega \\ 0 \end{pmatrix}(0 \;\; B_{c\omega}^T X_c) \equiv F_e + B_{e\omega}(0 \;\; B_{c\omega}^T X_c) \tag{5.46a}$$

$$H_{e\bar{\omega}} = H_e + \begin{pmatrix} 0 & 0 \\ 0 & B_{c\omega}^T X_c \end{pmatrix} \tag{5.46b}$$

$$H_{e\omega}^T H_{e\omega} = H_e^T H_e - \begin{pmatrix} 0 \\ X_c B_{c\omega} \end{pmatrix}(0 \;\; B_{c\omega}^T X_c). \tag{5.46c}$$

The following theorem gives the design:

Theorem 5.4. *With all assumptions and the decentralized design otherwise as in Theorem 4.4, let* $X \geq 0$ *satisfy*

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XS_{\bar{\Omega}} X + H^T H = 0, \tag{5.47}$$

and let $W > 0$ *satisfy*

$$\begin{aligned} WA_{c+}^T &+ A_{c+}W + \frac{1}{\alpha^2} WX_c B_c B_c^T X_c W - WC_c^T C_c W + G_c G_c^T \\ &+ \alpha^2 I_c S_\Omega I_c^T + (W - W_D) C_c^T C_c (W - W_D) = 0, \end{aligned} \tag{5.48}$$

where

$$I_c^T = [I \; I \ldots I]$$

$$A_{c+} = A_c + Diag(S_\Omega X, S_\Omega X, \ldots, S_\Omega X),$$

$$S_\Omega = B_\Omega B_\Omega^T,$$

$$S = S_\Omega + S_{\bar{\Omega}},$$

and $\Omega \subseteq \{1, 2, \ldots, q\}$. *Let the controllers be given by*

$$\dot{\xi}_i = (A + \frac{1}{\alpha^2} G_+ G_+^T X - SX - L_i C_i)\xi_i + L_i y_i, \quad i \in \{1, 2, \ldots, q\}, \tag{5.49a}$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\}, \tag{5.49b}$$

and assume all controllers are open-loop (internally) stable. Then, for controller outages corresponding with any $\omega \subseteq \Omega$, the closed-loop system (5.45) is internally stable, and the closed-loop transfer-function matrix $T_{\bar{\omega}}(s)$ from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$ satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$.

Remark 5.4: The design equations (5.47) and (5.48) arise from replacing G in the plant description (4.1) with the augmented matrix G_+ given by

$$G_+ = (G \;\alpha B_\Omega), \tag{5.50}$$

and changing the design equations accordingly. This substitution affects both the state-feedback design ARE and the Riccati-like design equation for computing decentralized observer gains. The substitution is equivalent to selecting P_e in (5.1) as

$$P_e = X_e \begin{pmatrix} S_\Omega & 0 \\ 0 & 0 \end{pmatrix} X_e \geq 0. \tag{5.51}$$

The basic design computed for the augmented plant will *provide reliable control* for the actual plant.

Proof: As in the development of Section 4.5, the existence of appropriate solutions to the design equations (5.47) and (5.48) guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_{e+} G_{e+}^T X_e + H_e^T H_e = 0. \tag{5.52}$$

Unlike the dual case, the additional columns of G_+ enter into the linear coefficient matrix F_e of (5.52), as well as into the quadratic coefficient as explicitly indicated. This is because the controller structure (5.49) is affected if G is replaced by G_+. Hence, F_e and G_{e+} are now given by

$$F_e = \begin{pmatrix} A & -BB_c^T X_c \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix}, G_{e+} = \begin{pmatrix} G_+ & 0 \\ 0 & L_c \end{pmatrix}, \tag{5.53}$$

with $A_{\alpha c} = \mathrm{Diag}(A_\alpha, A_\alpha, \ldots, A_\alpha)$ and $A_\alpha = A + \alpha^{-2} G_+ G_+^T X - SX$. Manipulations of (5.52) similar to those of the dual case, using (5.46), (5.50), and (5.53), give

$$\begin{aligned} F_{e\bar{\omega}}^T X_e \;&+\; X_e F_{e\bar{\omega}} + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e\bar{\omega}}^T H_{e\bar{\omega}} \\ &\leq \; -\left(X_e B_{e\omega} - \left(\begin{smallmatrix} 0 \\ X_c B_{c\omega} \end{smallmatrix}\right)\right)(B_{e\omega}^T X_e - (0 \; B_{c\omega}^T X_c) \leq 0. \end{aligned}$$

Provided $(F_{e\bar{\omega}}, H_{e\bar{\omega}})$ is a detectable pair, therefore, Lemma 4.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_{e\bar{\omega}}(sI - F_{e\bar{\omega}})^{-1}G_e$ satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$. To establish detectability, let $v^T = (v_1^T \; v_2^T) \neq 0$ satisfy $F_{e\bar{\omega}}v = \lambda v$ and $H_{e\bar{\omega}}v = 0$. Then $Av_1 = \lambda v_1$ and $Hv_1 = 0$. Since (A, H) is a detectable pair, this implies either $Re(\lambda) < 0$ or $v_1 = 0$. Suppose $v_1 = 0$; then $F_{e\bar{\omega}}v = \lambda v$ gives

$$(A_{\alpha c} - L_c C_c)v_2 = \lambda v_2. \tag{5.54}$$

Since all controllers are assumed open-loop stable, (5.54) gives $Re(\lambda) < 0$. □

The two decentralized designs given in Theorems 5.3 and 5.4 model controller failures as being, respectively, measurement failures and actuator failures. The failures considered incapacitate entire controllers, so that measurement failures and actuator failures have the same effect on the closed-loop transfer-function matrix. Although the two designs have the same reliability goals, they are nevertheless different: The first automatically guarantees reliable stability if the design equations have appropriate solutions, whereas the second may exist but not guarantee reliable stability if the controllers are not open-loop stable; the first design involves only modification of feedback and observer gains as compared with the basic design, while the second requires also a change in the observer structure; and the range of the design parameter a for which the two designs are computable may differ.

In the centralized case considered in Theorems 5.1 and 5.2, the failures considered are those of individual sensors or actuators. Therefore, the two centralized designs differ not only in the view taken of controller failure, and in other technical terms, but also in the reliability properties they seek to guarantee.

5.4 Example

For the example in Section 4.6 reliable designs have been computed for both the centralized and decentralized cases.

Centralized designs reliable with respect to an outage of each measurement were computed for several values of the design parameter α according to the procedure given in Theorem 5.1. Table 5.2 gives the actual H_∞ norms $\|T\|_\infty$ of the resulting closed-loop

Table 5.2: H_∞ norms for basic and reliable decentralized designs.

	Basic Design			Reliable Design 1			Reliable Design 2		
	no failure	y_1 fails	y_2 fails	no failure	y_1 fail	y_2 fails	no failure	y_1 fails	y_2 fails
$\alpha = 20$	3.09	unstab.	unstab.	4.38	2.90	2.95	4.34	4.68	4.26
$\alpha = 16$	3.09	unstab.	unstab.	4.26	2.78	2.88	4.29	4.46	4.11
$\alpha = 12$	3.06	unstab.	unstab.	4.06	2.61	2.78	4.21	4.13	3.89
$\alpha = 8$	3.01	unstab.	unstab.	3.76	2.38	2.76	4.12	3.55	3.46
$\alpha = 4$	2.72	34.28	unstab.	3.38	2.22	2.48	$\alpha^2 Y^{-1} - X \not> 0$		
$\alpha = 2$	1.95	4.41	unstab.	$\alpha^2 Y^{-1} - X \not> 0$			$\alpha^2 Y^{-1} - X \not> 0$		

transfer-function matrices. Results corresponding to the designs reliable with respect to outages of y_1 and y_2 are labelled "Reliable Design 1" and "Reliable Design 2," respectively. For each design, results are given corresponding to no sensor failure, failure of y_1, and failure of y_2. For the sake of comparison, Table 5.2, includes H_∞ norms corresponding to the basic centralized output-feedback design given in Theorem 4.3.

Reliable Design 1 theoretically guarantees stability and the closed-loop H_∞-norm bound α only for failure of y_1, and Reliable Design 2 only for failure of y_2; but in this example, each reliable design gives stability and the H_∞-norm bound in case of a failure of either measurement. In fact, a measurement failure would seem to result in a reduced H_∞ norm for the closed-loop system. This is so, however, because a measurement failure removes from consideration the corresponding measurement noise, effectively eliminating one column of the transfer-function matrix. More meaningful is a comparison of this (reduced) transfer-function matrix in the case of a failure with the corresponding transfer-function submatrix in the base case. Such a comparison shows that the H_∞ norm in case of a failure is larger than it is when no failure occurs. For example, Reliable Design 2 for $\alpha = 8$ results in the norms $\|T_2\|_\infty = 3.55$ and $\|T_1\|_\infty = 3.46$, as shown in Table 5.2, while the corresponding parts of the base-case transfer-function matrix (after elimination of the appropriate columns) have norms $\|T_2\|_\infty = 3.30$ and $\|T_1\|_\infty = 3.01$.

The reduced cost which can be achieved by eliminating measurements and their corresponding noise inputs does not constitute a valid argument for discarding one measurement and using the other alone. The use of both measurements provides reliability, in that a single sensor failure will not result in system instability.

Note that the solution of the observer-design ARE satisfies the condition $\alpha^2 Y^{-1} = X > 0$ only when $\alpha \geq 4$ for Reliable Design 1, or when $\alpha \geq 5$ (the case $\alpha = 5$ is not shown) for Reliable Design 2, while solutions were computed for the basic design with the design parameter value as small as $\alpha = 2$. this difference is an indication of the tradeoff between reliability and performance guaranteed by the respective designs.

For the same example, a decentralized control design reliable with respect to the failure of Control Channel 1 was computed for various values of the design parameter α according to the procedure of Theorem 5.3. Table 5.3 gives the actual H_∞ norms of the closed-

Table 5.3: H_∞ norms for basic and reliable decentralized designs.

	Basic Design			Reliable Design		
	no failure	#1 fails	#2 fails	no failure	#1 fails	#2 fails
$\alpha = 20$	3.63	unstable	5.34	6.95	6.25	7.03
$\alpha = 16$	3.63	unstable	5.30	7.65	6.38	7.82
$\alpha = 14$	3.61	unstable	5.28	8.28	6.32	8.59
$\alpha = 12$	3.59	unstable	5.23	No solution to RLAE found.		

loop transfer-function matrices resulting when the reliable design was computed for several values of α. Conditions corresponding with no control failure, with failure of Controller #1, and with failure of Controller #2 are given. For the sake of comparison, the comparison, the comparable portion of Table 5.3, corresponding with the basic decentralized design, is reproduced.

Table 5.3 shows that the reliable design guarantees stability and H_∞-norm bounds α in spite of a possible failure of Controller #1. As in the centralized case, the reduce H_∞ norm in case of a failure of Controller #1 results since, when Controller #1 fails, the disturbance w_1 and the control input u_1 are removed from consideration, eliminating one column and one row from the closed-loop transfer-function matrix. Again, this apparent reduction in cost does not constitute a valid argument for discarding Controller #1 and using Controller #2 alone, since the use of two controllers provides reliability, in that a single controller failure will not result in system instability. No solution was found to the Riccati-like algebraic equation for the reliable design with $\alpha \leq 13$, illustrating again the tradeoff between reliability and disturbance attenuation.

5.5 Strongly Stabilizing Designs

The designs given in Theorems 5.3 and 5.4 provide decentralized control laws which are reliable with respect to controller outages. For the design given in Theorem 5.3, all controllers susceptible to outages are automatically stable; however, for the design given in Theorem 5.4, the controllers must be assumed to turn out stable for the closed-loop system to be guaranteed stable. It is therefore of interest to develop designs which stabilize the plant via a stable control law. Such designs are referred to as "strongly stabilizing." Strong stabilization has been treated in [64], where a necessary and sufficient condition for the existence of a strongly stabilizing controller is given.

A decentralized design is now developed to guarantee open-loop stability of some subset of controllers, without regard for performance in case of a controller outage. This design may be combined with that of Theorem 5.4 so as to guarantee beforehand that specified controllers will turn out open-loop stable. As a special case, a strongly stabilizing centralized design is also derived.

With the design otherwise as in Theorem 4.4, suppose Equation (4.47) is replaced by

$$\begin{aligned} WA_c^T \quad &+ \quad \alpha_C W + \frac{1}{\alpha^2} WX_cB_cB_c^TX_cW - WC_c^TC_cW + G_cG_c^T \\ &+ \quad (W - W_D)C_c^TC_c(W - W_D) + P = 0. \end{aligned} \tag{5.55}$$

For any $P \geq 0$, the design guarantees closed-loop stability and the $H-\infty$-norm bound $\|T\|_\infty \leq \alpha$. The object is to select $P \geq 0$ so that the i^{th} controller is open-loop stable. Rewrite (5.55) as

$$W(A_c - L_cC_c)^T + (A_c - L_cC_c)W + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW + G_cG_c^T + L_cL_c^T + P = 0. \tag{5.56}$$

Recalling the definitions $A_{\alpha c} = \text{Diag}(A_\alpha, A_\alpha, \ldots, A_\alpha)$, $I_c^T = [I \ I \ldots I]$, and $A_c = A_{\alpha c} +$

$I_c B B_c^T X_c$, rewrite (5.56) as

$$\begin{aligned} &W(A_{\alpha c} - L_c C_c)^T + (A_{\alpha c} - L_c C_c)W + \frac{1}{\alpha^2} W X_c B_c B_c^T X_c W \\ &+ G_c G_c^T + L_c L_c^T + P + I_c B B_c^T X_c W + W X_c B_c B^T I_c^T = 0. \end{aligned} \tag{5.57}$$

The i^{th} $n \times n$ main-diagonal block of (5.57) is

$$\begin{aligned} &W_{ii}(A_\alpha - L_i C_i)^T + (A_\alpha - L_i C_i)W_{ii} + \frac{1}{\alpha^2}(W_{i1} \ldots W_{iq}) X_c B_c B_c^T X_c \begin{pmatrix} W_{1i} \\ \vdots \\ W_{qi} \end{pmatrix} \\ &+ G G^T + L_i L_i^T + P_{ii} + B B_c^T X_c \begin{pmatrix} W_{1i} \\ \vdots \\ W_{qi} \end{pmatrix} + (W_{i1} \ldots W_{iq}) X_c B_c B^T = 0, \end{aligned} \tag{5.58}$$

where the linear coefficient $(A_\alpha - L_i C_i)$ is the open-loop dynamic matrix of the i^{th} controller. To ensure that $(A_\alpha - L_i C_i)$ will be Hurwitz, let $P_{ii} = \alpha^2 S = \alpha^2 B B^T$. Then (5.58) becomes

$$\begin{aligned} &W_{ii}(A_\alpha - L_i C_i)^T + (A_\alpha - L_i C_i)W_{ii} + G G^T + L_i L_i^T \\ &= \left(-\alpha B + \frac{1}{\alpha}(W_{i1} \ldots W_{iq}) X_c B_c\right) \left(\alpha B^T + \frac{1}{\alpha} B_c^T X_c \begin{pmatrix} W_{1i} \\ \vdots \\ W_{qi} \end{pmatrix}\right) \le 0, \end{aligned} \tag{5.59}$$

with $W_{ii} > 0$. To see that this is sufficient to guarantee that $(A_\alpha - L_i C_i)$ is Hurwitz, let $v \neq 0$ satisfy $(A_\alpha - L_i C_i)^T v = \lambda v$. Then (5.59) gives

$$2Re(\lambda) v^* W_{ii} v + v^* L_i L_i^T v \le 0,$$

and hence $Re(\lambda) \le 0$. But inequality must hold here, because $Re(\lambda) = 0$ implies $L_i^T v = 0$, and hence $A_\alpha^T v = \lambda v$, with A_α assumed Hurwitz.

Note that $P_{ii} \ge \alpha^2 S$ guarantees that the i^{th} controller will be stable, independent of the other main-diagonal blocks of P. Therefore, several controllers may be simultaneously guaranteed open-loop stable by selecting the main-diagonal blocks P_{ii} of P to satisfy $P_{ii} \ge 0$ if the i^{th} controller need not be stable, and $P_{ii} \ge \alpha^2 S$ if the i^{th} controller must be stable. Other blocks of P may be chosen in any way that makes $P \ge 0$, such as setting them all to 0.

The following theorem summarizes the result.

Theorem 5.5. *Let P be any $qn \times qn$ matrix satisfying*

$$P \geq Diag(P_{11}, \ldots, P_{tt}, 0, \ldots, 0),$$

where $P_{ii} = \alpha^2 S$ for $i \in \{1, 2, \ldots, t\}$. With all assumptions and the design otherwise as in Theorem 4.4, suppose Equation (4.47) is replaced by

$$\begin{aligned} WA_c^T &+ A_cW + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW + G_cG_c^T \\ &+ (W - W_D)C_c^TC_c(W - W_D) + P = 0. \end{aligned} \tag{5.60}$$

Then the design, in addition to its other properties, guarantees that the controllers in the first t control channels are all open-loop stable.

The result of Theorem 5.5 is easily specialized to the centralized case. It is important to note, however, that the solution W of the Riccati-like design equation with $q = 1$ is not the same as the solution Y of the observer design ARE in the centralized case. Therefore, the reformulation of the design equations to guarantee strong stabilization in the centralized case is not as simple as that given in Theorem 5.5. The following theorem gives the correct formulation.

Theorem 5.6. *With all assumptions and the design otherwise as in Theorem 4.3, let $Y > 0$ satisfy the ARE*

$$YF^T + FY + \frac{1}{\alpha^2}YH^THY - YC^TCY + \frac{1}{\alpha^2}YXSXY + GG^T + \alpha^2S = 0, \tag{5.61}$$

where $F = A - SX$, $S = BB^T$. Then the system is strongly stable, and the closed-loop transfer-function matrix satisfies $\|T\|_\infty \leq \alpha$.

Proof. For the special case $q = 1$, the strong stabilization result of Theorem 5.5 still holds. In this case, the design equation (5.60) is

$$\begin{aligned} W(A + \alpha^{-2}GG^TX)^T &+ (A + \alpha^{-2}GG^TX)W + \frac{1}{\alpha^2}WXSXW \\ &- WC^TCW + GG^T + \alpha^2S = 0. \end{aligned} \tag{5.62}$$

Hence, the proof consists of showing that (5.61) implies (5.62). Recall the assumption from Theorem 4.3 that $\sigma_{\max}\{YX\} < \alpha^2$ or $(\alpha^2 Y^{-1} - X) > 0$. This implies that there exists a matrix $W > 0$ such that

$$W^{-1} = Y^{-1} - \alpha^{-2} X. \tag{5.63}$$

Then, routine manipulations of (5.61) give the equivalent equation

$$YA^T + AY + \frac{1}{\alpha^2} YH^T HY - YC^T CY + GG^T + \alpha^2 YW^{-1}SW^{-1}Y = 0. \tag{5.64}$$

Pre- and post-multiply (5.64) by Y^{-1}, and use (5.63) to obtain

$$\begin{aligned} A^T(\alpha^{-2}X + W^{-1}) &+ (\alpha^{-2}X + W^{-1})A + (\alpha^{-2}X + W^{-1})GG^T(\alpha^{-2}X + W^{-1}) \\ &+ \frac{1}{\alpha^2} H^T H - C^T C + \alpha^2 W^{-1} S W^{-1} = 0. \end{aligned} \tag{5.65}$$

Now, divide the state-feedback design ARE (4.3) by α^2 to obtain

$$A^T(\alpha^{-2}X) + (\alpha^{-2}X)A + (\alpha^{-2}X)GG^T(\alpha^{-2}X) - \frac{1}{\alpha^2} XSX + \frac{1}{\alpha^2} H^T H = 0, \tag{5.66}$$

and subtract (5.66) from (5.65) to obtain

$$\begin{aligned} A^T W^{-1} &+ W^{-1}A + (\alpha^{-2}X)GG^T W^{-1} + W^{-1}GG^T(\alpha^{-2}X) \\ &+ W^{-1}GG^T W^{-1} - C^T C + \frac{1}{\alpha^2} XSX + \alpha^2 W^{-1} S W^{-1} = 0. \end{aligned} \tag{5.67}$$

Finally, pre- and post-multiply (5.67) by W, and rearrange terms to obtain (5.62). □

6 EXTENSIONS

6.1 Robust Decentralized Control

The decentralized design methodology given in Section 4.7.3 is now extended to apply to a plant with structured (parametric) uncertainty. The resulting designs guarantee robust stability and an H_∞-norm bound for the closed-loop system, for any plant uncertainty in a bounded admissible set.

The results and methods applied to the study the robust control problems of interest here are closely connected to the topics of quadratic stability, and to H_∞-norm optimization. Among the relatively few papers that actually treat the problem of robust H_∞-bounding control in the presence of structured parametric uncertainty is [65], where a perturbations in (A, B, C) is represented as additional weighted noise inputs and measured outputs. The procedure suggested in [66] is then followed to solve the problem. As a consequence of the selected linear and quadratic bounding function a controller is defined via three coupled Riccati-like equations. The procedure developed here is in the same spirit, but with the restriction of plant variations in the A matrix only. The applied approach, which in essence implies a different bounding procedure, leads to two decoupled design equations.

The results are of interest is particular because they explain to the decentralized control case, but they also apply easily to the simpler state-feedback and centralized output-feedback cases, which are omitted here.

Structured uncertainty is introduced into the plant A-matrix according to the definition

$$A = A_0 + \sum_{k=1}^{r} G_k M_k H_k, \tag{6.1}$$

where A_0 is known, the G_k's and H_k's give the structure of the uncertainty, and each unknown constant matrix M_k satisfies

$$\lambda_{max}\{M_k M_k^T\} < \sigma_k^2. \tag{6.2}$$

If each positive bound σ_k is sufficiently small, then the design equations to be derived for robust control will have appropriate solutions. The existence of such solutions guarantees that the computed control law provides the desired robust stability and disturbance attenuation.

6.1.1 Robust design derivation

The derivation of the robust design methodology is similar to that of the reliable design methodology given in Section 5. The essential idea of the derivation is to formally express a sufficient condition for stability and disturbance attenuation for the nominal system, including a formal representation of some design freedom, and to determine how to use that freedom to guarantee robust stability and performance for the actual system.

The first step in the derivation is to fix the observer structure of the control law, and write the desired condition

$$F_{0e}^T X_e + X_e F_{0e} + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = 0, \tag{6.3}$$

where (F_{0e}, G_e, H_e) describes the nominal closed-loop system, and $P_e \geq 0$ is as yet unspecified. An appropriate value of P_e will be chosen to guarantee desired robustness properties. The plant uncertainty terms are omitted in (6.3), so that

$$F_{0e} = \begin{pmatrix} A_0 & -BB_c^T X_c \\ L_c C & A_{0\alpha c} - L_c C_c \end{pmatrix},$$

where A_0 represents the nominal plant dynamics, and the block-diagonal matrix $A_{0\alpha c} - L_c C_c$ represents the dynamics of the decentralized control law to be determined. Taking into account the plant uncertainty (6.1), the actual closed-loop dynamic matrix is given by

$$F_e = F_{0e} + \sum_{k=1}^{r} \begin{pmatrix} G_k \\ 0 \end{pmatrix} M_k (H_k \; 0) \equiv F_{0e} + \sum_{k=1}^{r} G_{ek} M_k H_{ek}. \tag{6.4}$$

Using (6.4), rewrite the condition (6.3) as

$$\begin{aligned} F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e &= -P_e + (F_e - F_{0e})^T X_e + X_e (F_e - F_{0e}) \\ &= -P_e + \sum_{k=1}^{r} \{ H_{ek}^T M_k^T G_{ek}^T X_e + X_e G_{ek} M_k H_{ek} \}. \end{aligned} \tag{6.5}$$

Now, $P_e \geq 0$ may be chosen such that the right-hand side of (6.5) is negative semi-definite. Recall that σ_k is given by (6.2), and set

$$P_e = \sum_{1=k}^{r} \{ H_{ek}^T H_{ek} + \sigma_k^2 X_e G_{ek} G_{ek}^T X_e \} \geq \sum_{k=1}^{r} \{ H_{ek}^T H_{ek} + X_e G_{ek} M_k M_k^T G_{ek}^T X e \}, \tag{6.6}$$

so that, after some manipulation, (6.5) gives

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e$$

$$= -\sum_{k=1}^{r} \{H_{ek}^T - X_e G_{ek} M_k\}\{H_{ek} - M_k^T G_{ek}^T X_e\} \tag{6.7}$$

$$-\sum_{k=1}^{r} \{X_e G_{ek}(\sigma_k^2 I - M_k M_k^T) G_{ek}^T X_e\} \leq 0.$$

Therefore, if $X_e \geq 0$ satisfies (6.3), with P_e given by (6.6), then the uncertain system satisfies the main hypothesis of Lemma 2.1.

The next step in the derivation of the robust control is to determine the needed modifications to the design equations (4.28) and (4.44) so that $X_e \geq 0$ satisfies (6.3), with P_e given by (6.6). By examination of (6.3) and (6.6), and of the definitions of G_e and H_e given in (4.34), it is easily seen that

$$\frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = \frac{1}{\alpha^2} X_e G_{e+} G_{e+}^T X_e + H_{e+}^T H_{e+},$$

where

$$G_{e+} = \begin{pmatrix} G_+ & 0 \\ 0 & L_c \end{pmatrix}, G_+ = (G \;\; \alpha\sigma_1 G_1 \;\; \ldots \;\; \alpha\sigma_r G_r), \tag{6.8a}$$

$$H_{e+} = \begin{pmatrix} H_+ & 0 \\ 0 & -B_c^T X_c \end{pmatrix}, H_+ = \begin{pmatrix} H \\ H_1 \\ \vdots \\ H_r \end{pmatrix}. \tag{6.8b}$$

Hence, the robust design is obtained by replacing the triple (A, G, H) with the triple (A_0, G_+, H_+) in the design equations (4.28) and (4.44) for the basic design. Using the augmented matrices G_+ and H_+ in the design equations is similar to introducing additional disturbance inputs and regulated outputs into the problem. Therefore, the smallest value of α for which the design equations will have a solution will be larger for the robust design than for the basic design.

Recall that, in the basic design, the controller dynamics depend on an assumed worst disturbance, and hence on the matrix G. Therefore, replacing G with G_+ in the design affects not only G_e, but also F_{0e}.

The final step in deriving the robust design is to establish that (F_e, H_e) is a detectable pair. Note that Lemma 4.2, applied to the modified design, establishes that (F_{0e}, H_{e+}) is a detectable pair, provided (A_0, H) is a detectable pair, $A_{0\alpha} \equiv A_{0+\alpha^{-2}} G_+ G_+^T X - SX$ is Hurwitz, and $A_{0\alpha} + SX$ has no $j\omega$-axis eigenvalues. Let $v \neq 0$ satisfy

$$F_e v = \lambda v, \quad H_e v = 0. \tag{6.9}$$

The detectability proof consists of proving that $\mathrm{Re}(\lambda) < 0$. Multiply (6.7) on the left by v^* and on the right by v to obtain

$$\begin{aligned} &2\mathrm{Re}(\lambda) v^* X_e v + \frac{1}{\alpha^2} v^* X_e G_e G_e^T X_e v + \sum_{k=1}^{r} v^* \{H_{ek}^T - X_e G_{ek} M_k\}\{H_{ek} - M_k^T G_{ek}^T X_e\} v \\ &\quad + \sum_{k=1}^{r} v^* \{X_e G_{ek}(\sigma_k^2 I - M_k M_k^T) G_{ek}^T X_e\} v \leq 0. \end{aligned} \tag{6.10}$$

Since every term in (6.10) but the first is nonnegative, this implies

$$\mathrm{Re}(\lambda) v^* X_e v \leq 0. \tag{6.11}$$

If inequality holds in (6.11), then $v^* X_e v > 0$ and $\mathrm{Re}(\lambda) < 0$. If equality holds, then every term in (6.10) is zero. This gives

$$\{H_{ek} - M_k^T G_{ek}^T X_e\} v = 0, \quad k \in \{1, 2, \ldots, r\}. \tag{6.12}$$

But, since (6.2) implies $\sigma_k^2 I - M_k M_k^T$ is nonsingular, (6.10) also gives $G_{ek}^T X_e v = 0$, so that (6.12) gives

$$H_{ek} v = 0, \quad k \in \{1, 2, \ldots, r\}. \tag{6.13}$$

Hence, (6.4) and (6.9) give $F_{0e} v = \lambda v$, while (6.8b) and (6.9) give $H_{e+}^T H_{e+} v = 0$. Since (F_{0e}, H_{e+}) is a detectable pair, this implies $\mathrm{Re}(\lambda) < 0$.

Theorem 6.1 summarizes the result. The following definitions are convenient:

$$A_{0\alpha c} = \text{Diag}\,(A_{0\alpha}, A_{0\alpha}, \ldots, A_{0\alpha}), \tag{6.14a}$$

$$A_{0\alpha} = A_0 + \frac{1}{\alpha^2} G_+ G_+^T X - SX, \tag{6.14b}$$

$$A_{0c} = A_{0\alpha c} + I_c B B_c^T X_c, \tag{6.14c}$$

$$G_{c+} = I_c G_+. \tag{6.14d}$$

$$I_c^T = [I \; I \; \ldots \; I] \in \mathbb{R}^{n \times qn}.$$

Theorem 6.1. *Suppose the plant (2.1), with decentralized control structure given by (4.26), has constant structured uncertainty (6.1), with*

$$\lambda_{\max}\{M_k M_k^T\} < \sigma_k, \quad k \in \{1, 2, \ldots, r\}.$$

Define G_+ and H_+ as in (6.8), and let $X \geq 0$ satisfy

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X G_+ G_+^T X - X S X + H_+^T H_+ = 0, \tag{6.15}$$

and $W > 0$ satisfy the Riccati-like algebraic equation

$$\begin{aligned} &W A_{0c}^T + A_{0c} W + \frac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W \\ &+ G_{c+} G_{c+}^T + (W - W_D) C_c^T C_c (W - W_D) = 0. \end{aligned} \tag{6.16}$$

Suppose also that (A_0, H) is a detectable pair, $A_{0\alpha}$ is Hurwitz, and $A_{0\alpha} + SX$ has no eigenvalues on the $j\omega$-axis. Then the decentralized control law

$$\dot{\xi}_i = (A_{0\alpha} - L_i C_i)\xi_i + L_i y_i, \quad i \in \{1, 2, \ldots, q\},$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\},$$

with $L_i = W_{ii} C_i^T$, $i \in \{1, 2, \ldots, q\}$, robustly stabilizes the uncertain plant, and the closed-loop transfer-function matrix $T(s)$ from w_e to z satisfies

$$\|T\|_\infty \leq \alpha.$$

There is no explicit restriction on the size of the bounds σ_k in Theorem 6.1. However, the larger the σ_k's are taken to be, the larger α will need to be to obtain solutions to the design equations (6.15) and (6.16); if the σ_k's are taken to be too large, no solutions may exist at all. If bounds σ_k on the size of the uncertainty are known accurately, then these bounds should be incorporated in G_+ (or H_+), and hence in the design equations. If the resulting design equations can be solved, then the design can tolerate uncertainties of the specified size. On the other hand, if uncertainty bounds are not accurately known, the choice of the σ_k's may be used to reflect a relative weighting to be given by the design to disturbance attenuation and robustness considerations. Since changing the values of the σ_k's in this case is equivalent to rescaling the G_k's and H_k's while holding the σ_k's fixed, it may simplify the design procedure to set

$$\sigma_k = \frac{1}{\alpha}, \quad k \in \{1, 2, \ldots, r\},$$

and scale the G_k's and H_k's with respect to G and H so as to reflect the tradeoff between robustness and disturbance attenuation. Then, the size of the uncertainty which may be tolerated is determined indirectly by finding the smallest value of a for which the design equations can be solved. This variation on the above design is given in Theorem 6.2.

Theorem 6.2. *Suppose the plant (2.1), with decentralized control structure defined by (4.26), has constant structured uncertainty (6.1), with*

$$\lambda_{\max}\{M_k M_k^T\} < \frac{1}{\alpha^2}, \quad k \in \{1, 2, \ldots, r\}.$$

Define $G_+ = (G\ G_1\ \ldots G_r)$ *and* $H_+^T = \left(H^T\ H_1^T \ldots H_r^T\right)$. *Let* $X \geq 0$ *satisfy (6.15) and let* $W > 0$ *satisfy the Riccati-like algebraic equation (6.16). Suppose also that* (A_0, H) *is a detectable pair,* $A_{0\alpha}$ *is Hurwitz, and* $A_{0\alpha} + SX$ *has no eigenvalues on the* $j\omega$*-axis. Then the decentralized control law*

$$\dot{\xi}_i = (A_{0\alpha} - L_i C_i)\xi_i + L_i y_i, \quad i \in \{1, 2, \ldots, q\},$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\},$$

with $L_i = W_{ii} C_i^T$, $i \in \{1, 2, \ldots, q\}$, *robustly stabilizes the uncertain plant, and the closed-loop transfer-function matrix* $T(s)$ *from* w_e *to* z *satisfies*

$$\|T\|_\infty \leq \alpha.$$

6.1.2 Example

This section presents an example of robust state-feedback control design. The example illustrates the difference between the robust designs of Theorems 6.1 and 6.2, and the use of the parameter σ_1 to determine the largest uncertainty in a certain class for which the design guarantees stability and the predetermined H_∞-norm bound. For these purposes, a state-feedback example is adequate, and has the advantage of avoiding the complication of decentralized design, already studied in Section 4.

Consider nominal plant

$$A_0 = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} G = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} H = (1\ 0\ -1\ 0),$$

and introduce the structured uncertainty

$$A = A_0 + G_1 M_1 H_1,$$

where M_1 is an unknown scalar, and G_1 and H_1 are given by

$$G_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, H_1 = (0\ 0\ 1\ 0). \tag{6.17}$$

This represents an uncertainty in the (4,3) element of the A-matrix of the plant. As in the decentralized design of Theorem 6.1, the robust state-feedback control is found by doing a basic design, but with the augmented matrices G_+ and H_+ in place of G and H, where in this case

$$G_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \alpha\sigma_1 \end{pmatrix}, H_+ = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The state-feedback design equation becomes

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X G_+ G_+^T X - XSX + H_+^T H_+ = 0,$$

or equivalently

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X G G^T X + \sigma_1^2 X G_1 G_1^T X - X S X + H^T H + H_1^T H_1 = 0. \tag{6.18}$$

In the second quadratic term of (6.18), the α's cancel out, allowing computation of a solution for $\alpha = \infty$. By setting $\alpha = \infty$ and solving (6.18) with various values of σ_1, one may determine a largest plant perturbation (corresponding with $|M_1| = \sigma_{1_{\max}}$) for which at least stability can be guaranteed using the robust state-feedback design. Then, given any $\sigma_1 \leq \sigma_{1_{\max}}$, one may determine a number $\alpha_{\min}$ such that for any $\alpha \geq \alpha_{\min}$ there exists an appropriate solution of (6.18), and therefore an associated design guaranteeing the robust H_∞-norm bound $\|T\|_\infty \geq \alpha$ for the closed-loop system. Table 6.1 gives the values of $\alpha_{\min}$, to the nearest 0.1, computed for various values of σ_1, and shows clearly the tradeoff between robustness and optimal disturbance rejection. In this example, the largest admissible plant perturbation is given approximately by $\sigma_{1_{\max}} = 1.8$.

Table 6.1: Approximate minimum H_∞-norm bounds for various plant uncertainties.

σ_1	1.0	1.2	1.4	1.6	1.8
$\alpha_{\min}$	1.4	1.5	1.6	1.8	3.0

If $\sigma_1 = \alpha^{-1}$, as in Theorem 6.2, then for G_1 and H_1 given by (6.17) the design equation becomes

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X(G G^T + G_1 G_1^T) X - X S X + H^T H + H_1^T H_1 = 0. \tag{6.19}$$

The approximate smallest value of α for which (6.19) has an appropriate solution is $\alpha_{\min} = 1.4$, which corresponds with a plant uncertainty bound $\sigma_1 = 0.71$.

6.2 Computation of Families of H_∞ Control Laws

6.2.1 Introduction

Given a set of control design requirements, a designer may wish to characterize a family of control laws which satisfy these requirements. The characterization of a family of such admissible control laws may permit the selection of a particular controller with additional desirable properties.

Youla *et al.* [67] give a parameterization of all stabilizing controllers in terms of stable coprime factorizations of the plant and a baseline stabilizing controller. The parameter space consists of the set of all stable proper transfer-function matrices Q(s) of appropriate dimensions. Doyle *et al.* [1] and Glover and Doyle [47] give a parameterization of all stabilizing controllers which provide the H_∞-norm bound $\|T\|_\infty \leq \alpha$ for the closed-loop transfer-function matrix $T(s)$. The parameter space consists of the set of all stable proper transfer-function matrices $Q(s)$ of appropriate dimensions satisfying $\|Q\|_\infty \leq \alpha$. The controller parameterizations given in [67], [1], and [47] have the advantage of spanning the set of all controllers with the desired (stabilization or disturbance-attenuation) properties. Unfortunately, they have two substantial disadvantages: First, they include controllers of arbitrarily high order; second, they do not retain the structural properties of the baseline controller, such as strict properness or decentralized control structure. Restricting the order of the parameter Q(s) also bounds the order of the controller; however, structural properties of the baseline controller are still not retained.

Based on the parameterization of Youla, [68] gives a characterization of all stabilizing decentralized control laws. The approach is simply to restrict the Youla parameter $Q(s)$ to values which give control laws with block-diagonal structure. This restriction is shown to be equivalent to a set of matrix algebraic constraints on the parameter. Unfortunately, there is no clear way of selecting the parameter to satisfy these constraints. Even if appropriate parameters could be found, the corresponding controllers would still not retain the order or strict properness of the baseline controller.

This section gives a characterization of families of H_∞-suboptimal control laws starting from known baseline designs. The derivation of the baseline controllers in each case has been accomplished in Section 4 by fixing a controller structure and selecting controller parameters such that a certain algebraic Riccati equation (ARE), with closed-loop system matrices as coefficients, has a positive semi-definite solution. In this paper, families of controllers are derived by retaining the baseline controller structure and finding controller parameters such that a relaxed sufficient condition, in the form of an algebraic Riccati inequality, is satisfied. Suitable controller parameters are found by exploiting a convexity property of a matrix Riccati function.

First, a family of state-feedback gains all of which guarantee the same H_∞-norm bound is given. Then, a family of observer-based output-feedback controllers is given, based on some member of the state-feedback family. Each member of the output-feedback family has the same observer structure; therefore, each is strictly proper and is of the same order as the plant. Finally, a family of decentralized control laws is derived in the decentralized case, where the baseline controller contains a full-order observer of the plant in each control channel. Again, the (strictly proper) observer structure and the controller order are retained by each member of the family.

No claim is made that the families of controllers given here include every controller with the desired characteristics; however, the families are easily computed, and exclude all non-strictly proper and high-order controllers.

6.2.2 The matrix Riccati function

This section gives some properties of a matrix Riccati function related to the computation of families of H_∞-suboptimal control laws. The matrix Riccati function is studied in greater detail in [10].

Define the matrix Riccati function R on the space of symmetric matrices by

$$R(X) = F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H. \tag{6.20}$$

Then the following property is easily verified:

Lemma 6.1. *Suppose F is Hurwitz and $R(X) = 0$. Then $X \geq 0$.*

Proof. Write out $R(X) = 0$ as

$$R(X) \equiv F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H = 0. \tag{6.21}$$

Define $P = \alpha^{-2}XGG^TX + H^TH$. Then (6.21) becomes $F^TX + XF + P = 0$, with F Hurwitz and $P \geq 0$. By inertia theorems of the Lyaunov equation (see, for example, [20], $X \geq 0$ □

The following lemma gives a matrix convexity property for R. This property does not require that F be Hurwitz, but only that the quadratic coefficient be positive semi-definite.

Lemma 6.2. *For $i \in \{1, \ldots, r\}$, let X_i be symmetric matrices and β_i be nonnegative scalars satisfying $\Sigma_{i=1}^r \beta_i = 1$. Then*

$$R\left\{\sum_{i=1}^r \beta_i X_i)\right\} \leq \sum_{i=1}^r \beta_i R(X_i). \tag{6.22}$$

Proof. Compute

$$R\left\{\sum_{i=1}^r \beta_i X_i\right\} = F^T\left\{\sum_{i=1}^r \beta_i X_i\right\} + \left\{\sum_{i=1}^r \beta_i X_i\right\}F + \frac{1}{\alpha^2}\left\{\sum_{i=1}^r \beta_i X_i\right\}GG^T\left\{\sum_{i=1}^r \beta_i X_i\right\} + H^TH$$

$$= \sum_{i=1}^r \beta_i(F^TX_i + X_iF + H^TH) + \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^r \beta_i\beta_j X_i GG^T X_j$$

$$= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r \beta_i X_i GG^T X_i\left\{\sum_{j=1}^r \beta_j\right\} + \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^r \beta_i\beta_j X_i GG^T X_j$$

$$= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^r \beta_i\beta_j X_i GG^T(X_i - X_j)$$

$$= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j X_i GG^T(X_i - X_j)$$

$$+\frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1+i}^r \beta_i\beta_j X_i GG^T(X_j - X_i)$$

$$= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^i\sum_{j=1}^{i-1} \beta_i\beta_j X_i GG^T(X_i - X_j)$$

$$+\frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j X_j GG^T(X_i - X_j)$$

$$= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j (X_i - X_j)GG^T(X_i - X_j). \tag{6.23}$$

Therefore, $R\left\{\Sigma_{i=1}^{r}\beta_i R(X_i)\right\} \leq \sum_{i=1}^{r}\beta_i R(X_i)$, the desired result. $\square$

The following corollary identifies a class of easily computable matrices $Z \geq 0$ for which $R(Z) \leq 0$:

Lemma 6.3. *Let Z be any convex combination of matrices $X_i \geq 0$, $i \in \{1, 2, \ldots, r\}$, satisfying $R(X_i) = 0$.*

Then $Z \geq 0$ satisfies

$$R(Z) \leq 0. \tag{6.24}$$

Proof: Express Z as

$$Z = \sum_{i=1}^{r}\beta_i X_i,$$

where $\Sigma_{i=1}^{r}\beta_i = 1$. From Lemma 6.1 $Z \geq 0$, and from Lemma 6.2,

$$R(Z) = R\left\{\sum_{i=1}^{r}\beta_i X_i\right\} \leq \sum_{i=1}^{r}\beta_i R(X_i) = 0.$$

$\square$

6.2.3 A family of state-feedback controls

Note that, for the state-feedback case, any matrix $X \geq 0$ satisfying (4.8) is suitable for computing the control $u = -B^T X x$. In fact, this control would still be suitable if the left-hand side of (4.8) were negative semi-definite; that is, any control law given by

$$u = -B^T Z x, \tag{6.25}$$

$$A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H \leq 0, \quad Z \geq 0 \tag{6.26}$$

provides stability and the H_∞-norm bound a for the closed-loop system.

A given solution $X \geq 0$ of (4.8) will be called the "baseline" solution. Given one such solution, a family of matrices $Z \geq 0$ satisfying (6.26), and hence a family of stabilizing state-feedback control laws which guarantee the closed-loop bound $\|T\|_\infty \leq \alpha$, is characterized.

Take $X \geq 0$ to be the baseline solution of (4.8). Given this fixed matrix X, define the matrix Riccati function R by

$$R(M) = F^T M + MF + \frac{1}{\alpha^2} MGG^T M + (XBB^T X + H^T H), \tag{6.27}$$

where $F = A - BB^T X$ is Hurwitz. By Lemma 6.1, each symmetric solution of $R(X_i) = 0$ satisfies $X_i \geq 0$. Let $Z \geq 0$ be any convex combination of solutions X_i of $R(X_i) = 0$. By Lemma 6.3,

$$R(Z) \equiv F^T Z + ZF + \frac{1}{\alpha^2} ZGG^T Z + (XBB^T X + H^T H) \leq 0. \tag{6.28}$$

To see that $Z \geq 0$ satisfies (6.26), rearrange (6.28) to obtain

$$\begin{aligned} & A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H \\ & \leq -ZBB^T Z + ZBB^T X + XBB^T Z - XBB^T X \\ & = -(Z - X)BB^T(Z - X) \leq 0. \end{aligned}$$

The following theorem summarizes the characterization of a family of state-feedback H_∞ controls:

Theorem 6.3. *Suppose $F = A - BB^T X$, where $X \geq 0$ solves the state-feedback design ARE (4.8). Let Z be any convex combination of solutions X_i of the ARE*

$$F^T X_i + X_i F + \frac{1}{\alpha^2} X_i GG^T X_i + (XBB^T X + H^T H) = 0. \tag{6.29}$$

Then, $F_Z = A - BB^T Z$ is Hurwitz, and the state-feedback control law

$$u = -B^T Z x$$

guarantees that

$$T(s) = \begin{pmatrix} H \\ -B^T Z \end{pmatrix} (sI - F_Z)^{-1} G$$

satisfies $\|T\|_\infty \leq \alpha$.

6.2.4 A family of output-feedback controls

The approach of Theorem 6.3 extends to the output-feedback case: Start with $Z \geq 0$ a convex combination of solutions X_i of (6.29). Define

$$N_1 = A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H. \tag{6.30}$$

By Lemma 6.3, $N_1 \leq 0$. The following theorem now gives a family of observers for each state-feedback H_∞ control characterized by such a Z.

Theorem 6.4. *Assume $A + \alpha^{-2} GG^T Z - BB^T Z$ is Hurwitz. Let $Y > 0$ satisfy*

$$AY + YA^T + \frac{1}{\alpha^2} YH^T HY - YC^T CY + GG^T = 0, \tag{6.31}$$

with $(A - YC^T C)$ Hurwitz. Let $V > 0$ be any convex combination of solutions Y_i of

$$(A - YC^T C)Y_i + Y_i(A - YC^T C)^T + \frac{1}{\alpha^2} Y_i(H^T H - N_1)Y_i + (YC^T CY + GG^T) = 0 \tag{6.32}$$

satisfying $\sigma_{\max}\{VZ\} < \alpha^2$, and define the observer gain L by

$$L = (I - \alpha^{-2} VZ)^{-1} VC^T = (V^{-1} - \alpha^{-2} Z)^{-1} C^T. \tag{6.33}$$

Then, the controller

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2} GG^T Z - BB^T Z - LC \right) \xi + Ly, \tag{6.34a}$$

$$u = -B^T Z \xi, \tag{6.34b}$$

stabilizes the plant (2.1), and provides the closed-loop H_∞-norm bound $\|T\|_\infty \leq \alpha$.

Proof. First note that, since $N_1 \leq 0$, $(H^T H - N_1) \geq 0$. By Lemma 6.3,

$$(A - YC^T C)V + V(A - YC^T C)^T + \frac{1}{\alpha^2} V(H^T H - N_1)V + (YC^T CY + GG^T) \leq 0. \tag{6.35}$$

Algebraic manipulations similar to those in the proof of Theorem 3.1 give

$$AV + VA^T + \frac{1}{\alpha^2} VH^T HV - VC^T CV + GG^T \leq \frac{1}{\alpha^2} VN_1 V. \tag{6.36}$$

Pre- and post-multiply (6.36) by αV^{-1} to obtain

$$(\alpha^2 V^{-1}) + A^T(\alpha^2 V^{-1}) + H^T H - \alpha^2 C^T C + \frac{1}{\alpha^2}(\alpha^2 V^{-1})GG^T(\alpha^2 V^{-1}) \leq N_1. \tag{6.37}$$

Subtract (6.30) from (6.37) to obtain

$$\begin{aligned} &(\alpha^2 V^{-1} - Z)A + A^T(\alpha^2 V^{-1} - Z) - \alpha^2 C^T C \\ &+\frac{1}{\alpha^2}(\alpha^2 V^{-1})GG^T(\alpha^2 V^{-1}) + ZBB^T Z - \frac{1}{\alpha^2} ZGG^T Z \leq 0. \end{aligned} \tag{6.38}$$

Define

$$X_1 = (\alpha^2 V^{-1} - Z) > 0, \tag{6.39}$$

and rewrite (6.38) as

$$X_1 A + A^T X_1 - \alpha^2 C^T C + \frac{1}{\alpha^2}(X_1 + Z)GG^T(X_1 + Z) + ZBB^T Z - \frac{1}{\alpha^2} ZGG^T Z \leq 0. \tag{6.40}$$

Now define $N_2 \leq 0$ as the left-hand side of (6.40); rearranging terms, (6.40) becomes

$$\begin{aligned} N_2 \equiv\ & X_1(A + \alpha^{-2}GG^T Z - LC) + (A + \alpha^{-2}GG^T Z - LC)X_1 \\ &+\alpha^2 C^T C + \frac{1}{\alpha^2} X_1 GG^T X_1 + ZBB^T Z \leq 0. \end{aligned} \tag{6.41}$$

With the controller (6.34), the closed-loop system transformed to error coordinates is described by

$$F_e = \begin{pmatrix} A - BB^T Z & -BB^T Z \\ \alpha^{-2}GG^T Z & A + \alpha^{-2}GG^T Z - LC \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ -G & L \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ -B^T Z & -B^T Z \end{pmatrix}.$$

Define

$$X_e = \begin{pmatrix} Z & 0 \\ 0 & X_1 \end{pmatrix} \geq 0,$$

and consider the quantity

$$X_e F_e + F_e^T X_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e. \tag{6.42}$$

It can now be verified that the two off-diagonal blocks of (6.42) are identically zero, and that the diagonal blocks give N_1 and N_2, as defined in (6.30) and (6.41); therefore,

$$X_e F_e + F_e^T X_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \leq 0.$$

The proof of detectability of (F_e, H_e) is routine, and proceeds exactly as that in Section 4.4.

Therefore, by Lemma 4.1, F_e is Hurwitz, and $T(s) = H_e(sI - F_e)^{-1}G_e$ satisfies $\|T\|_\infty \leq \alpha$.

□

Recall that [48] and [47] give parameterizations of the set of all output-feedback controllers guaranteeing the H_∞-norm bound α. Some of these controllers are of high order, and are therefore undesirable. By contrast, Theorem 6.4 characterizes a family of controllers with realizations all of the same order as the plant, which all guarantee the H_∞-norm bound α.

6.2.5 A family of decentralized controls

A generalization of Theorem 6.4 to the decentralized case cannot be readily obtained. Manipulations like those in the proof of Theorem 6.4 applied to the Riccati-like (decentralized) design equation do not give the desired result. Therefore, while Theorem 6.4 gives a family of observer designs for each state-feedback design, the next theorem gives only one decentralized observer design for each state-feedback design of Theorem 6.3. The definitions of Z and N_1 assumed in the theorem statement are as in Section 6.2.4. For the remainder of this section, every occurrence of X in (4.35) is replaced by Z.

Theorem 6.5. *Assume $A + \alpha^{-2}GG^TZ - BB^TZ$ is Hurwitz and $A + \alpha^{-2}GG^TZ$ has no $j\omega$-axis eigenvalues. Let $W > 0$ satisfy the Riccati-like algebraic equation*

$$A_cW + WA_c^T + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW + G_cG_c^T + (W - W_D)C_c^TC_c(W - W_D) = 0, \tag{6.43}$$

and compute $L_c = Diag(L_1, L_2, \ldots, L_q)$ as

$$L_c = W_DC_c^T. \tag{6.44}$$

Then, the decentralized control law

$$\dot{\xi}_i = (A + \frac{1}{\alpha^2}GG^TZ - BB^TZ - L_iC_i)\xi_i + L_iy_i, \quad i \in \{1, 2, \ldots, q\}, \tag{6.45a}$$

$$u_i = -B_i^TZ\xi_i, \quad i \in \{1, 2, \ldots, q\}, \tag{6.45b}$$

stabilizes the plant (2.1), with decentralized control structure given by (4.26), and provides the closed-loop H_∞-norm bound $\|T\|_\infty \le \alpha$.

Proof: Using (6.44), rewrite (6.43) as

$$(A_c - L_cC_c)W + W(A_c - L_cC_c)^T + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW + G_cG_c^T + L_cL_c^T = 0. \tag{6.46}$$

Pre- and post-multiply (6.46) by αW^{-1} to obtain

$$\begin{aligned}&(\alpha^2W^{-1})(A_c - L_cC_c) + (A_c - L_cC_c)(\alpha^2W^{-1}) + X_cB_cB_c^TXc\\ &+\frac{1}{\alpha^2}(\alpha^2W^{-1})(G_cG_c^T + L_cL_c^T)(\alpha^2W^{-1}) = 0.\end{aligned} \tag{6.47}$$

With controllers (6.45), the closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A - BB^TZ & -BB_c^TZ_c \\ \alpha^{-2}G_cG^TZ & A_c - L_cC_c \end{pmatrix}, G_e = \begin{pmatrix} G & 0 \\ -G_c & L_c \end{pmatrix}, H_e = \begin{pmatrix} H & 0 \\ -B^TZ & -B_c^TZ_c \end{pmatrix}, \tag{6.48}$$

where (6.48) differs from (4.36) only in that X has been replaced everywhere by Z. Define

$$X_e = \begin{pmatrix} Z & 0 \\ 0 & \alpha^2W^{-1} \end{pmatrix} \ge 0,$$

and consider the quantity

$$X_eF_e + F_e^TX_e + \frac{1}{\alpha^2}X_eG_eG_e^TX_e + H_e^TH_e. \tag{6.49}$$

The two off-diagonal blocks of (6.49) are identically zero. The upper-left block of (6.49) gives N_1 defined in (6.30). The lower-right block is zero by (6.47). Therefore,

$$X_eF_e + F_e^TX_e + \frac{1}{\alpha^2}X_eG_eH_e^TX_e + H_e^TH_e = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix} \le 0.$$

By Lemma 4.2, (F_e, H_e) is a detectable pair; therefore, by Lemma 4.1, the closed-loop system is stable, and the closed-loop transfer-function matrix $T(s) = H_e(sI - F_e)^{-1}G_e$ satisfies $\|T\|_\infty \le \alpha$. □

Similar to Theorem 6.4 in the centralized case, Theorem 6.5 gives a family of decentralized control laws which guarantee a predetermined H_∞-norm bound for the closed-loop system, and which are characterized by controllers of the same order as the plant. Unlike the centralized case, the family of decentralized controls consists of only a single controller associated with each member of a family of state-feedback controls.

6.2.6 Conclusions

A convexity property of a certain matrix Riccati function is used to characterize families of controllers which provide stability and H_∞ disturbance attenuation. This characterization has two significant advantages over those given [67], [48], and [47]: First, it includes families of decentralized control laws. Second, it excludes controllers of high order. It is possible that some controllers in the families developed here could have realizations of lower order than that of the corresponding baseline controller; hence, a criterion for choosing among the controllers could be the order of their minimal realizations. How to choose from the family a controller with a lower-order minimal realization is a problem for future research.

6.3 H-Infinity Control in Discrete Time

In this section we develop state-feedback control laws that provide disturbance attenuation with a uniform H_∞-norm bound for discrete systems using state-feedback and output-feedback controls and also discuss methods of computing H_∞ norms of discrete-time systems. For completeness, we include known results regarding H_∞-norm bounds of discrete-time systems and results on characterizing state-feedback control laws that guarantee certain H_∞ norm bounds.

6.3.1 Preliminary results

We consider the system

$$x_{k+1} = Ax_k + Bu_k + Gw_k \tag{6.50a}$$

$$\zeta_k = \begin{bmatrix} Hx_k \\ u_k \end{bmatrix} \tag{6.50b}$$

where u is the control input, ζ is the regulated output and w is a square-summable disturbance input. We also make the assumption that A is invertible, and that the triple $[A, B, H]$ is stabilizable and detectable. Our development will make constant reference to the discrete-time algebraic Riccati equation (DARE) of the general form

$$P = R_1 + A^T P(I + R_2 P)^{-1} A \tag{6.51}$$

where, in general, R_1 and R_2 are symmetric but not necessarily sign-definite, as opposed to LQ control with $R_1 \leq 0$ and $R_2 > 0$. We associate with this equation, the symplectic matrix

$$\$ = \begin{bmatrix} A + R_2 A^{-T} R_1 & -R_2 A^{-T} \\ -A^{-T} R_1 & A^{-T} \end{bmatrix}.$$

The following Lemma relates the stabilizing solution of (6.51) with the eigenvalues of \$ [69].

Lemma 6.4. *If \$ has no eigenvalues on the unit circle and $[A, R_2]$ is stabilizable, then equation (6.51) has a unique stabilizing solution P (i.e., such that the spectrum of $A - R_2 A^{-T}(P - R_1)$ lies in the open unit disk.)* □

The results developed here are directly related to that of finding a controller that achieves desired disturbance attenuation for the discrete system (6.50). This problem can be posed as one of choosing a control strategy that minimizes cost functional

$$J(u) = \sup \left\{ \frac{\|\zeta\|_2}{\|w\|_2} : w \in \ell_2, w \neq 0 \right\}. \tag{6.52}$$

Alternatively, we may define the cost functional

$$V = \|\zeta\|_2^2 - \gamma^2 \|w\|_2^2.$$

Then $V \leq 0$ for all $w \in l_2$, if and only if $\|T(z)\|_\infty < \gamma$, where $T(z)$ the transfer function from w to ζ. In this case, we say that the system has a disturbance attenuation of γ. It has been shown in [40] that the minimizing control is a linear function of the state. We may therefore restrict consideration to linear closed-loop systems. The functional (6.52) is then equivalent to the H_∞ norm of the system, and we may, therefore, formulate the optimal disturbance rejection problem as follows: Determine the stabilizing state-feedback control $u = K_{\min} x$ such that for all stabilizing K

$$\|T_{\min}\|_\infty \leq \|T_c\|_\infty$$

where

$$T_{\min}(z) = \begin{bmatrix} H \\ K_{\min} \end{bmatrix} (zI - A - BK_{\min})^{-1} G, \qquad T_c(z) = \begin{bmatrix} H \\ K \end{bmatrix} (zI - A - BK)^{-1} G.$$

Note that the H_∞ norm is considered here to be the "sup norm" of essentially bounded functions on the unit circle with analytic extension to the region outside the unit disk. The next result identifies conditions under which this may be accomplished using state feedback [40].

Theorem 6.6. *Suppose that the triple $[A, B, H]$ is stabilizable and detectable, and that there exists a stabilizing solution P of*

$$P = H^T H + A^T P \left[I + (BB^T - \frac{1}{\gamma^2} GG^T) P \right]^{-1} A \tag{6.53}$$

with $\gamma > 0$ and the side condition

$$\gamma^2 I - G^T PG > 0. \tag{6.54}$$

Then the control $u = Kx$, with

$$K = -B^T P \left[I + (BB^T - \frac{1}{\gamma^2} GG^T) P \right]^{-1} A \tag{6.55}$$

guarantees that

a)the matrix $(A + BK)$ is stable,

b)the matrix $[A + (BB^T - \frac{1}{\gamma^2} GG^T) A^{-T} (P - H^T H)]$ is stable, and

c) $\|T_c\|_\infty < \gamma$, where $T_c(z) = \begin{bmatrix} H \\ K \end{bmatrix} (zI - A - BK)^{-1} G$.

□

A related result is found in [70]. In the setting of "perfect information," used in [70], however, the control was allowed to depend on both the state and the disturbance. The disturbance, however, is generally not known.

The results of [40] show that the optimal solution to the disturbance-attenuation problem exists as a state-feedback control. That is, consider a decreasing sequence $\{\gamma_i, i \in \mathcal{N}\}$ such that for each γ_i, conditions (6.53) and (6.54) hold. This sequence, being bounded away from zero, will have a limit, which we call $\gamma_{\min}$. The controller gain K given in (6.55) also has a limit [40], $K_{\min}$, and we have

$$\gamma_{\min} = \|T_{\min}\|_\infty, \tag{6.56}$$

where

$$T_{\min}(z) = \begin{bmatrix} H \\ K_{\min} \end{bmatrix} (zI - A - BK_{\min})^{-1} G. \tag{6.57}$$

In the sequel, $\gamma_{\min}$ will denote the minimal achievable H_∞-norm bound using state feedback.

The following Lemma, which is very similar to the previous result, is useful in designing suboptimal H_∞-norm-bounding controllers.

Lemma 6.5. *Suppose $[E, H]$ is detectable and for some $\gamma > 0$, $\delta > 0$, and $P \geq 0$ we have*

$$P \geq \frac{1}{\delta^2} H^T H + E^T P \left(I - \frac{1}{\gamma^2} G G^T P \right)^{-1} E \tag{6.58}$$

$$\gamma^2 I - G^T P G > 0. \tag{6.59}$$

Then for $T(z) = H(zI - E)^{-1} G$

a)E is stable, and

b)$\|T\|_\infty \leq \gamma\delta$.

Proof. For completeness we prove this result; it follows in spirit the continuous case and the case where (6.58) is an equality.

a) Condition (6.58) implies that there exists $N \leq 0$ such that

$$N = \frac{1}{\delta^2} H^T H + E^T P E - P + M^T S^{-1} M \tag{6.60}$$

where $M = G^T P E$, and $S = \gamma^2 I - G^T P G$. Using condition (6.59), the fact that $P \geq 0$, and the detectability of (E, H), we verify the stability of E by standard results on the discrete Lyapunov equation.

b) Let

$$\begin{aligned} x_{k+1} &= E x_k + G w_k, \qquad x_o = 0, w \in l_2 \\ \zeta_k &= H x_k. \end{aligned} \tag{6.61}$$

Define

$$V_\delta = \frac{1}{\delta^2}\|\zeta\|_2^2 - \gamma^2\|w\|_2^2. \tag{6.62}$$

We are done if we can show that $V_\delta \le 0$ for all $w \in l_2$, since this will imply that

$$\|\zeta\|_2^2 - \gamma^2\delta^2 \le \|w\|_2^2 \le 0 \tag{6.63}$$

for all $w \in l_2$, thus giving the attenuation bound. Since E is stable, and $x_o = 0$

$$x_o^T P x_o = -\sum_{k=0}^{\infty}(x_{k+1}^T P x_{k+1} - x_k^T P x_k) = 0; \tag{6.64}$$

thus, using (6.64)

$$\begin{aligned}
V_\delta &= V_\delta - x_o^T P x_o \\
&= \sum_{k=0}^{\infty}\frac{1}{\delta^2}\zeta_k^T\zeta_k - \gamma^2 w_k^T w_k + x_{k+1}^T P x_{k+1} - x_k^T P x_k \\
&= \sum_{k=0}^{\infty}\frac{1}{\delta^2}(Hx_k)^T(Hx_k) - \gamma^2 w_k^T w_k + (Ex_k + Gw_k)^T P(Ex_k + Gw_k) - x_k^T P x_k \\
&= \sum_{k=0}^{\infty} x_k^T\left\{\frac{1}{\delta^2}H^T H + E^T P E - P\right\}x_k - w_k^T\{\gamma^2 I - G^T P G\}w_k + 2w_k^T G^T P E x_k \\
&= \sum_{k=0}^{\infty} x_k^T\{N - M^T S^{-1} M\}x_k - w_k^T\{S\}w_k + 2w_k^T M x_k
\end{aligned}$$

by (6.58). Let $w_k^* = S^{-1}Mx_k$, introduce $w_k^\circ = w_k - w_k^*$ and note that

$$\begin{aligned}
w_k^T S w_k &= [w_k - w_k^*]^T S[w_k - w_k^*] + 2w_k^T S w_k^* - w_k^{*T} S w_k^* \\
&= w_k^{\circ T} S w_k^\circ + 2w_k^T S w_k^* - w_k^{*T} S w_k^\circ.
\end{aligned}$$

Then V_δ becomes

$$\begin{aligned}
V_\delta &= \sum_{k=0}^{\infty} x_k^T N x_k - \{w_k^{\circ T} S w_k^\circ + 2w_k^T S w_k^* - w_k^{*T} S w_k^*\} + x_k^T M w_k^* + 2x_k^T M^T w_k \\
&= \sum_{k=0}^{\infty} x_k^T N x_k - w_k^{\circ T} S w_k^\circ + 2w_k^T(Mx_k - Sw_k^*) - w_k^{*T}(Mx_k - Sw_k^*) \\
&= \sum_{k=0}^{\infty} x_k^T N x_k - w_k^{\circ T} S w_k^\circ \le 0
\end{aligned}$$

for all $w \in l_2$, since $w_k^* = S^{-1}Mx_k$, $N \leq 0$ and $S > 0$. □

Thus, if we can find any $P \geq 0$ that satisfies the conditions of Lemma 6.5, then the stability of the system is guaranteed, and the uniform H_∞-norm bound holds. If instead of the inequality we have equality, then for $\delta = 1$ we have the discrete equivalent of [71]:

Corollary 6.1. *Suppose $[E, H]$ is detectable and for some $P \geq 0$ and $\gamma > 0$ we have*

i)

$$P = H^T H + E^T P \left(I - \frac{1}{\gamma^2} G G^T P \right)^{-1} E$$

ii)

$$\gamma^2 I - G^T P G > 0.$$

Then for $T(z) = H(zI - E)^{-1}G$

a) E is stable, and

b)$\|T\|_\infty < \gamma$.

□

Remark 6.1. If $P > 0$ then we can remove the detectability condition.

Corollary 6.1 provides an iterative means of computing a tight upper bound on the H_∞ norm, with each successive iteration providing a better estimate of the actual value. To determine the H_∞ norm, we choose $\gamma > 0$, and then test to see if the conditions of Corollary 6.1 are met. This procedure is then repeated with a lower value of γ to find a new upper bound. Determining the actual H_∞ norm involves a search for γ_s, where

$$\gamma_s =: \inf \left\{ \gamma > 0 : \exists p \geq 0 \text{ such that } P = H^T H + E^T P \left(I - \frac{1}{\gamma^2} G G^T P \right)^{-1} E \geq 0 \right.$$
$$\left. \text{and} \gamma^2 I - G^T P G > 0 \right\}$$

so that

$$\|T\|_\infty = \|H(zI - E)^{-1}G\|_\infty = \gamma_s.$$

This is fully analogous to the algorithms proposed for the continuous case [71]. A more efficient algorithm for computing the H_∞ norm in the discrete case is described in Section 6.3.4.

We also note for future reference the following two well known facts.

Lemma 6.6. *If E is stable and $P = E^T PE + S$, for some $S \geq 0$, then $P \geq 0$.*

Proof. Since E is stable then we can write $P = \Sigma_{i=0}^{\infty}(E^T)^k SE^k \geq 0$. □

Lemma 6.7. *If (A, H) is detectable, then for any gain K, $(E, \bar{H})$ is detectable, where $E = A + BK$, and $\bar{H} = [H^T \;\; K^T]^T$.*

Proof. Let λ be an eigenvalue of E corresponding to an unobservable mode of $(E, \bar{H})$; that is, there exists a vector $v \neq 0$ such that

$$Ev = \lambda v \tag{6.65a}$$

and

$$\bar{H}v = 0. \tag{6.65b}$$

The proof consists in showing that $|\lambda| < 1$. Equation (6.65b) implies that $Hv = 0$, and $Kv = 0$, which in turn implies that $Ev = (A + BK)v = Av = \lambda v$, and thus corresponds to an unobservable mode of (A, H). Since (A, H) is detectable, $|\lambda| < 1$. □

Thus, detectability is not lost under the proposed state-feedback structure and augmented output matrix.

6.3.2 Properties of Riccati operators

We now introduce a discrete-time Riccati operator, analogous to that for the continuous problem introduced in [10], establish its convexity properties, and proceed to find a set of state-feedback laws that guarantee H_∞-norm bounds. The general approach is based on the discrete Algebraic Riccati inequality (DARI) of Lemma 6.5.

For a fixed, positive scalar γ, and matrices E, G, H, and $\bar{H}$, we introduce over the field $S = \{X \in \mathbb{R}^{n\times n} : X = X^T\}$ the two domains

$$S_I = \{X \in S : (\gamma^2 I - G^T XG)^{-1} \text{ exists}\} \tag{6.66}$$

$$S_p = \{X \in S : (\gamma^2 I - G^T X G) > 0\}, \tag{6.67}$$

and define the operators $DR : S_I \to S$, and $R : S_I \to S$ where

$$\begin{aligned} DR(X) &= \bar{H}^T \bar{H} - X + E^T X \left[I - \frac{1}{\gamma^2} G G^T X \right]^{-1} E \\ &= \bar{H}^T \bar{H} - X + E^T \left[I - \frac{1}{\gamma^2} X G G^T \right]^{-1} X E \end{aligned} \tag{6.68}$$

and

$$R(X) = H^T H - X + A^T X \left[I + \left(B B^T - \frac{1}{\gamma^2} G G^T \right) X \right]^{-1} A. \tag{6.69}$$

Note that when $U = \frac{1}{\gamma^2} G G^T$, $E = F_s$, and $\bar{H} = [H^T K_s^T]$, where

$$K_s = -B^T X_s \left[I + \left(B B^T - \frac{1}{\gamma^2} G G^T \right) X_s \right]^{-1} A, \tag{6.70}$$

$$F_s = A + B K_s, \tag{6.71}$$

and X_s is the stabilizing solution of $R(X) = 0$, then

$$DR(X) = H^T H + K_s^T K_s + F_s^T X (I - UX)^{-1} F_s - X \tag{6.72}$$

$$R(X) = H^T H + K_x^T K_x + F_x^T X (I - UX)^{-1} F_x - X \tag{6.73}$$

where

$$K_x = -B^T X \left[I + \left(B B^T - \frac{1}{\gamma^2} G G^T \right) X \right]^{-1} A \tag{6.74}$$

$$F_x = A + B K_x. \tag{6.75}$$

Note also that if $R(X) = 0$ then X solves equation (6.53), and similarly, for this choice of E and $\bar{H}, DR(X_s) = R(X_s)$.

We first explore general properties of the two Riccati operators and then proceed to define and demostrate a convexity property of the Riccati operator $DR(X)$. The first property relates to the monotonicity of the stabilizing solution of the Riccati equation $R(P) = 0$ with respect to γ. The next three properties relate to the solutions of $DR(M) = 0$.

As usual, given $X(\gamma) \in R^{n \times m}$, the matrix $\frac{\partial \mathbf{X}}{\partial \gamma}$ is defined if all the partials exist.

Theorem 6.7. *If $\gamma > \gamma_{\min}$ and P is the stabilizing solution of*

$$P = H^T H + A^T P \left[I + \left(BB^T - \frac{1}{\gamma^2} GG^T \right) P \right]^{-1} A,$$

then $\dfrac{\partial P}{\partial \gamma}$ exists and is a negative semidefinite matrix.

Proof. Let $\gamma_1 = \gamma + \epsilon$ for $\epsilon > 0$, where without loss of generality we can let ϵ be small enough that $\gamma_2 = \gamma - \epsilon > \gamma_{\min}$. Let $\alpha_1 = 1/\gamma_1^2$ and $\alpha_2 = 1/\gamma_2^2$. and let P_1 and P_2 be the stabilizing solutions of

$$\begin{aligned} P_1 &= H^T H + A^T P_1 [I + (BB^T - \alpha_1 GG^T) P_1]^{-1} A \\ P_2 &= H^T H + A^T P_2 [I + (BB^T - \alpha_2 GG^T) P_2]^{-1} A. \end{aligned}$$

Then

$$\begin{aligned} \Delta P &= P_1 - P_2 \\ &= A^T \{ P_1 [I + (BB^T - \alpha_1 GG^T) P_1]^{-1} - [I + P_2 (BB^T - \alpha_2 GG^T)]^{-1} P_2 \} A \\ &= A^T [I + P_2 (BB^T - \alpha_2 GG^T)]^{-1} \{ [I + P_2 (BB^T - \alpha_2 GG^T)] P_1 - P_2 [I + (BB^T - \alpha_1 GG^T) P_1] \} \\ &\quad \times [I + (BB^T - \alpha_1 GG^T) P_1]^{-1} A \\ &= E_2^T \Delta P E_1 + \Delta \alpha D_2^T D_1 \end{aligned}$$

where $\Delta \alpha = \alpha_1 - \alpha_2$, $E_i = [I + (BB^T - \alpha_i GG^T) P_i]^{-1} A$, $D_i = G^T E_i$, for $i = 1, 2$. Thus

$$\frac{\Delta P}{\Delta \alpha} = E_2^T \left(\frac{\Delta P}{\Delta \alpha} \right) E_1 + D_2^T D_1.$$

Taking the limit as $\epsilon \to 0$, we have $P_1 \to P$, $P_2 \to P$, $E_1 \to E$, $E_2 \to E$, $D_1 \to D$ and $D_2 \to D$; therefore,

$$\frac{\partial P}{\partial \alpha} = E^T \left(\frac{\partial P}{\partial \alpha} \right) E + D^T D.$$

But by Theorem 6.6, E is stable and so by Lemma 6.6, $\dfrac{\partial P}{\partial \alpha} \geq 0$ and thus

$$\frac{\partial P}{\partial \gamma} = \frac{\partial P}{\partial \alpha} \frac{d\alpha}{d\gamma} = \frac{\partial P}{\partial \alpha} \left(-\frac{2}{\gamma^3} \right) \leq 0. \qquad \square$$

Remark 6.2: Theorem 6.7 tells us that if $\gamma_1 > \gamma_2 > \gamma_{\min}$, then $P(\gamma_1) \leq P(\gamma_2) \leq P(\gamma_{\min})$. This result could be strengthened to strict monotonicity if controllability conditions are imposed.

Remark 6.3: If P is chosen as the anti-stabilizing solution, i.e., E^{-1} is stable, then $P(\gamma)$ is monotonically increasing, i.e., for $\gamma_1 > \gamma_2 > \gamma_{\min}$, then $P(\gamma_1) \geq P(\gamma_2) \geq P(\gamma_{\min})$.

Lemma 6.8. *If E is a stability matrix, $M \in S_p$ and $DR(M) = 0$, then $M \geq 0$.*

Proof: Suppose $DR(M) = 0$ i.e.,

$$DR(M) = H^T H - M + E^T M E + E^T M G(\gamma^2 I - G^T M G)^{-1} G^T M E = 0;$$

thus,

$$E^T M E - M = -H^T H - E^T M G(\gamma^2 I - G^T M G)^{-1} G^T M E \leq 0.$$

Since E is a stability matrix, then by Lemma 6.6, $M \geq 0$. □

Theorem 6.8. *Suppose M_s and M_a are the "stabilizing" and "anti-stabilizing" solutions of $DR(M) = 0$. If M is any other matrix with $DR(M) = 0$, then*

$$M_s \leq M \leq M_a.$$

Proof: M_s is the stabilizing solution if $F_{os} = \left(I - \dfrac{1}{\gamma^2} G G^T M_s\right)^{-1} E$ has $\rho(F_{os}) < 1$. Under this condition, we have

$$E^T M_s \left(I - \frac{1}{\gamma^2} G G^T M_s\right)^{-1} E = F_{os}^T \left(I - \frac{1}{\gamma^2} M_s G G^T\right) M_s F_{os}.$$

Introduce again $U = \dfrac{1}{\gamma^2} G G^T$; then

$$DR(M_s) = 0 = H^T H + F_{os}^T M_s F_{os} - M_s - F_{os}^T M_s U M_s F_{os}. \tag{6.76}$$

Note also that the expression below can be manipulated as follows:

$$\begin{aligned}
& E^T M(I - UM)^{-1} E - F_{os}^T M F_{os} \\
&= E^T \left\{(M(I - UM)^{-1} - (I - M_s U)^{-1} M (I - U M_s)^{-1}\right\} E \\
&= E^T (I - M_s U)^{-1} \{(I - M_s U) M (I - UM)^{-1} (I - U M_s) - M\} (I - U M_s)^{-1} E \\
&= E^T (I - M_s U)^{-1} \Big\{(M - M_s U M_s + (M - M_s) G(\gamma^2 I - G^T M G)^{-1} \\
&\qquad G^T (M - M_s) - M)\Big\} (I - U M_s)^{-1} E
\end{aligned}$$

and so

$$E^T M(I-UM)^{-1}E - F_{os}^T M F_{os}$$
$$= F_{os}^T \left\{(M-M_s)G(\gamma^2 I - G^T MG)^{-1}G^T(M-M_s) - M_s U M_s s\right\} F_{os}. \tag{6.77}$$

Thus, since M is a solution, we have in analogy with (6.76), and using (6.77)

$$\begin{aligned} DR(M) = 0 &= H^T H + E^T M(I-UM)^{-1}E - M \\ &= H^T H + F_{os}^T M F_{os} - M + E^T M(I-UM)^{-1}E - F_{os}^T M F_{os} \\ &= H^T H + F_{os}^T M F_{os} - M - F_{os}^T M_s U M_s F_{os} \\ &\quad + F_{os}^T(M-M_s)G(\gamma^2 I - G^T MG)^{-1}G^T(M-M_s)F_{os} \\ &= DR(M_s) + F_{os}^T(M-M_s)F_{os} + M_s - M \\ &\quad + F_{os}^T(M-M_s)G(\gamma^2 I - G^T MG)^{-1}G^T(M-M_s)F_{os}, \end{aligned}$$

and so

$$\begin{aligned} 0 &= DR(M) - DR(M_s) \\ &= F_{os}^T(M-M_s)F_{os} - (M-M_s) \\ &\quad + F_{os}^T(M-M_s)G(\gamma^2 I - G^T MG)^{-1}G^T(M-M_s)F_{os}. \end{aligned}$$

Since F_{os} is stable, and $F_{os}^T(M-M_s)F_{os} - (M-M_s) \leq 0$, Lemma 6.6 gives $M - M_s \geq 0$ or $M_s \leq M$. Similarly, for the upper bound, we can write, (with $\rho(F_{0a}^{-1}) < 1$)

$$\begin{aligned} DR(M_a) &= H^T H + F_{oa}^T M_a F_{oa} - M_a - F_{oa}^T M_a U M_a F_{oa} \\ DR(M) &= H^T H + F_{oa}^T M F_{oa} - M - F_{oa}^T M_a U M_a F_{oa} \\ &\quad + F_{oa}^T(M_a - M)G(\gamma^2 I - G^T MG)^{-1}G^T(M_a - M)F_{oa}. \end{aligned}$$

and so

$$\begin{aligned} 0 &= DR(M) - DR(M_a) \\ &= (M_a - M) - F_{oa}^T(M_a - M)F_{oa} + F_{oa}^T(M_a - M)G(\gamma^2 I - G^T MG)^{-1}G^T(M_a - M)F_{oa}. \end{aligned}$$

Premultiplying by F_{oa}^{-T}, postmultiplying by F_{oa}^{-1} and rearranging gives

$$F_{oa}^{-T}(M_a - M)F_{oa}^{-1} - (M_a - M) = -(M_a - M)G(\gamma^2 I - G^T MG)^{-1}G^T(M_a - M) \leq 0.$$

Since F_{oa}^{-1} is stable, then $M_a \geq M$. □

Theorem 6.9. *If $M \in S_I$, (E,H) is detectable, and $DR(M) = 0$, then $M \in S_p$ (i.e., $\gamma^2 I - G^T MG > 0$).*

Proof. Suppose $M \geq H^T H$, and let $W = E^{-T}\{M - H^T H\}E^{-1} \geq 0$. Thus

$$DR(M) = 0 = -E^T W E + E^T M[I - UM]^{-1} E$$

which implies that $M[I - UM]^{-1} \geq 0$. But

$$M[I - UM]^{-1} = [I - MU]^{-1} M^{1/2}\{(I - M^{1/2}UM^{1/2})M^{1/2}[I - UM]^{-1}$$

which is positive semidefinite if

$$\gamma^2(I - M^{1/2}UM^{1/2}) \geq 0.$$

This is equivalent to

$$\gamma^2 I - G^T MG \geq 0.$$

But since $M \in S_I$, then $(\gamma^2 I - G^T MG)$ has no zero eigenvalues, and so $\gamma^2 I - G^T MG > 0$. We now show that $M \geq H^T H$. Since (E,H) is a detectable pair, then there exists $\bar{M} \geq 0$ such that

$$\bar{M} = H^T H + E^T \bar{M} E$$

which implies that $\bar{M} \geq H^T H$. Note also that as $\gamma \to \infty$ in $DR(M) = 0$, that $M(\gamma) \to \bar{M}$. Recall also that γ_s is the infimal number such that for all $\gamma \in (\gamma_\infty, \infty)$ there exist real solutions of $DR(M) = 0$. By Theorem 6.7, $M_s(\gamma)$ is monotonically decreasing for $\gamma \in (\gamma_s, \infty)$, and so

$$M_s(\gamma) \geq M_s(\infty) = \bar{M} \geq H^T H \geq 0.$$

By Theorem 6.8, we verify that $M_u(\gamma) \geq M(\gamma) \geq M_s(\gamma) \geq H^T H$. □

Definition 6.1. *A set* **A** *is called* convex *if and only if x_1, $x_2 \in$* **A**, *and $\alpha \in [0,1]$, implies that $\alpha x_1 + (1-\alpha)x_2 \in$* **A**.

Note that S_I is not a convex set, but S_p is convex.

Definition 6.2. *Suppose $f : \mathbb{R}^{n\times n} \rightarrow \mathbb{R}^{n\times n}$, $f(x) = f(x)^T$, for all x on $\mathbf{A}$, which is a convex set, then f is convex on $\mathbf{A}$ if and only if for all $x_1, x_2 \in \mathbf{A}$, and $\alpha \in [0,1]$,*

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2) \tag{6.78}$$

where "$\leq$" is defined in the sign-definite sense.

Theorem 6.10. *DR is convex on S_p.*

The proof of this property will require a few preliminary results on DR and its Fréchet differential [72]. The first Fréchet differential $\delta DR(x;\Delta)$ exists on S_I and is given in the following Lemma.

Lemma 6.9. *$\delta DR(x;\Delta) = -\Delta + E^T[I - xU]^{-1}\Delta[I - Ux]^{-1}E$ on S_I, where $U = \dfrac{1}{\gamma^2}GG^T$.*

Proof. We must show that for x an interior point of S_I, and for Δ such that $x + \Delta$ belongs to a neighborhood of x in S_I, that

$$\lim_{\|\Delta\|\to 0} \frac{\|DR(x+\Delta) - DR(x) - \delta DR(x;\Delta)\|}{\|\Delta\|} = 0. \tag{6.79}$$

We may, without loss of generality, restrict Δ such that for some $0 < M_1 < \infty$

$$\|[I - (x+\Delta)U]^{-1}\| \leq M_1\|[I - xU]^{-1}\|. \tag{6.80}$$

Let $M_2 = \|E\|^2\|[I - Ux]^{-1}\|^3\|U\|$. We now have

$$\begin{aligned}
&DR(x+\Delta) - DR(x) - \delta DR(x;\Delta)\\
&\quad = E^T\left\{(x+\Delta)[I - U(x+\Delta)]^{-1} - [I - xU]^{-1}x\right\}E - E^T[I - xU]^{-1}\Delta[I - Ux]^{-1}E\\
&\quad = E^T[I - xU]^{-1}\left\{(I - xU)(x+\Delta) - x(I - U(x+\Delta))\right\}[I - U(x+\Delta)]^{-1}E\\
&\qquad -E^T[I - xU]^{-1}\Delta[I - Ux]^{-1}E\\
&\quad = E^T[I - xU]^{-1}\Delta\left\{[I - U(x+\Delta)]^{-1} - [I - Ux]^{-1}\right\}E\\
&\quad = E^T[I - xU]^{-1}\Delta[I - U(x+\Delta)]^{-1}U\Delta[I - Ux]^{-1}E.
\end{aligned}$$

Thus

$$\begin{aligned}
\|DR(x+\Delta) - DR(x) - \delta DR(x;\Delta)\| &\leq M_1\|E\|^2\|[I - Ux]^{-1}\|^3\|U\|\|\Delta\|^2\\
&= M_1M_2\|\Delta\|^2
\end{aligned}$$

and so we readily see that

$$\lim_{\|\Delta\|\to 0} \frac{\|DR(x+\Delta) - DR(x) - \delta DR(x;\Delta)\|}{\|\Delta\|} \leq \lim_{\|\Delta\|\to 0} M_1 M_2 \|\Delta\| = 0. \tag{6.81}$$

□

Remark 6.4: Note that $\delta DR(x;\Delta)$ is linear in Δ; that is, for a scalar $\alpha, \delta DR(x;\alpha\Delta) = \alpha\delta DR(x;\Delta)$, and also $\delta DR(x;\Delta_1 + d\Delta_2) = \delta DR(x;\Delta_1) + \delta DR(x;d\Delta_2)$.

Remark 6.5: Since $\delta DR(x;\Delta)$ exists, it can also be shown to equal

$$\delta DR(x;\Delta) = \lim_{\alpha\to 0} \frac{DR(x+\alpha\Delta) - DR(x)}{\alpha}$$

So, not only does the first Fréchet differential exist for DR on S_I, but actually the Taylor series may be found. Remark 6.4 demonstrates that the derivative can be considered as a linear operator on its domain [73]. We may thus write $\delta DR(x;\Delta)$ as $DR_x(\Delta)$.

Although the convexity considered in this section is in the sign-definite sense, rather than element-wise, the following result is established in the same way as for scalar convex functions of a vector variable. The proof given here is a generalization of that given in [74] and relates convexity with the first Fréchet differential.

Lemma 6.10. *DR is convex on S_p iff for all $X, Y \in S_p$*

$$DR(Y) - DR(X) \geq DR_x(Y - X). \tag{6.82}$$

Proof: Suppose DR is convex on S_p. Then for all $\alpha \in [0,1]$ and X, $Y \in S_p$

$$DR(\alpha Y - (1-\alpha)X)) \leq \alpha DR(Y) + (1-\alpha)DR(X).$$

Now for $\alpha \in (0,1]$, we have

$$\frac{DR(X + \alpha(Y - X) - DR(X)}{\alpha} \leq DR(Y) - DR(X).$$

Letting $\alpha \to 0$ we obtain (6.82), by Remark 6.5.

Now let $X, Y \in S_p$, and $\alpha \in [0, 1]$, and let $Z = \alpha X + (1-\alpha)Y$. Then $Z \in S_p$ and

$$DR(X) \geq DR(Z) + DR_z(X - Z) \tag{6.83}$$

$$DR(Y) \geq DR(Z) + DR_z(Y - Z). \tag{6.84}$$

Multiplying (6.83) by α and (6.84) by $(1-\alpha)$ and adding gives

$$\alpha DR(X) + (1-\alpha)DR(Y) \geq DR(Z) + \alpha DR_z(X - Z) + (1-\alpha)DR_z(Y - Z). \tag{6.85}$$

By the definition of Z and Remark 6.4,

$$\begin{aligned}
\alpha DR_z(X - Z) + (1-\alpha)DR_z(Y - Z) &= DR_z[\alpha(X - Z)] + (1-\alpha)DR_z(Y - Z) \\
&= DR_z[(1-\alpha)(Z - Y)] + (1-\alpha)DR_z(Y - Z) \\
&= -(1-\alpha)DR_z(Y - Z) + (1-\alpha)DR_z(Y - Z) \\
&= 0.
\end{aligned}$$

Substituting $Z = \alpha X + (1-\alpha)Y$ into (6.85), then gives

$$\alpha DR(X) + (1-\alpha)DR(Y) \geq DR(\alpha X + (1-\alpha)Y)$$

and so $DR(X)$ is convex.

□

Proof of Theorem 6.10: Based on Lemma 6.10, DR is convex on S_p if for all X, $Y \in S_p$,

$$DR(Y) - DR(X) \geq DR_x(Y - X)$$

so this is what is proved. Let X, $Y \in S_p$; then

$$DR(Y) - DR(X) = -(Y - X) + E^T\{Y(I - UY)^{-1} - X(I - UX)^{-1}\}E.$$

Now

$$\begin{aligned}
(I - XU)Y(I - UY)^{-1}(I - UX) &= (I - XU)Y[I + (I - UY)^{-1}U(Y - X)] \\
&= (I - XU)Y + [I + (Y - X)U(I - YU)^{-1}]YU(Y - X) \\
&= Y - XUY + YU(Y - X) + (Y - X)U(I - YU)^{-1}YU(Y - X).
\end{aligned}$$

Thus

$$Y(I-UY)^{-1} - X(I-UX)^{-1}$$

$$= (I-XU)^{-1}\{(I-XU)Y(I-UY)^{-1}(I-UX) - (I-XU)X\}(I-UX)^{-1}$$

$$= (I-XU)^{-1}\{(Y-X) + (Y-X)U(Y-X)$$

$$+(Y-X)U(I-YU)^{-1}YU(Y-X)\}(I-UX)^{-1}$$

$$= (I-XU)^{-1}\{(Y-X) + (Y-X)U(I-YU)^{-1}(Y-X)\}(I-UX)^{-1},$$

and so

$$DR(Y) - DR(X) = -(Y-X)$$

$$+E^T(I-XU)^{-1}\{(Y-X) + (Y-X)U(I-YU)^{-1}(Y-X)\}(I-UX)^{-1}E.$$

On the other hand

$$DR_x(Y-X) = -(Y-X) + E^T(I-XU)^{-1}(Y-X)(I-UX)^{-1}E.$$

Subtracting gives

$$DR(Y) - DR(X) - DR_x(Y-X)$$

$$= E^T(I-XU)^{-1}(Y-X)G(\gamma^2 I - G^T YG)^{-1}G^T(Y-X)(I-UX)^{-1}E \geq 0.$$

□

The next Corollary, states the convexity property in terms of a convex combination of elements in S_p.

Corollary 6.2. *If $X_i \in S_p$ for $i = 1, 2, \ldots, q$, and $\beta_i \geq 0$ with $\beta_1 + \ldots + \beta_q = 1$, then*

$$DR\left(\sum_{i=1}^{q}\beta_i X_i\right) \leq \sum_{i=1}^{q}\beta_i DR(X_i). \tag{6.86}$$

□

We next define the admissible set of candidate feedback solutions. To this end let

$$S_\gamma = \left\{X = \sum_{i=1}^{q}\beta_i X_i : X_i \in S_I, DR(X_i) = 0, \quad \beta_i \geq 0, \quad \sum_{i=1}^{q}\beta_i = 1\right\} \tag{6.87}$$

where $DR(\cdot)$ is the operator (6.68) with $E = F_s = A + BK_s$ $\bar{H}^T = [H^T K_s^T]^T$, and K_s is given by (6.55), and formed from the stabilizing solution of the Riccati equation (6.53). Note that based on Lemma 6.8 and Theorems 6.8 to 6.10 it follows that $S_p \subset S_\gamma$. Note also that X_s the stabilizing solution of $DR(X) = 0$, is the "least" element of S_γ (i.e., $X_s \leq X$ for all $X \in S_\gamma$).

Lemma 6.11. *If $X \in S_\gamma$, then $DR(X) \leq 0$.*

Proof. Recall that $X = \sum_{i=1}^{q} \beta_i X_i$, with $\beta_1 + \ldots + \beta_q = 1$. Thus by Corollary 6.2,

$$DR(X) \leq \sum_{i=1}^{q} \beta_i DR(X_i) = 0. \qquad \square$$

Lemma 6.12. *If $X \in S_\gamma$, then $R(X) \leq 0$.*

Proof. By Lemma 6.11. $0 \geq DR(X)$, and so we only need to show that $DR(X) \geq R(X)$. We have

$$\begin{aligned} DR(X) - R(X) &= K_s^T K_s - K_x^T K_x + F_s^T X[I - UX]^{-1} F_s - F_x^T X[I - UX]^{-1} F_x \\ &= T(X, X_s). \end{aligned} \tag{6.88}$$

Let

$$M(X) = X^{-1} + BB^T - U, \tag{6.89}$$

$$N(X) = M(X) - BB^T. \tag{6.90}$$

Note that $N(X) > 0$, $M(X) > 0$. Then

$$K_x^T K_x + F_x^T X[I - UX]^{-1} F_x = A^T M(X)^{-1} A \tag{6.91}$$

and similarly

$$K_s^T K_s + F_s^T X[I - UX]^{-1} F_s = A^T M(X)^{-1} \{BB^T + N(X_s) N(X)^{-1} N(X_s\} M(X_s)^{-1} A. \tag{6.92}$$

So based on (6.91), (6.92), with $M(x)$, $N(x)$ defined by (6.89), (6.90) we find that

$$\begin{aligned} T(X, X_s) &= A^T \{M(X_s)^{-1} [BB^T + N(X_s) N(X)^{-1} N(X_s)] M(X_s)^{-1} - M(X)^{-1}\} A \\ &= A^T M(X_s)^{-1} \{BB^T + N(X_s) N(X)^{-1} N(X_s) - M(X_s) M(X)^{-1} M(X_s)\} M(X_s)^{-1} A \\ &= A^T M(X_s)^{-1} \{Q(X, X_s\} M(X_s)^{-1} A \end{aligned} \tag{6.93}$$

with

$$\begin{aligned} Q(X, X_s) &= BB^T + N(X_s) N(X)^{-1} N(X_s) - M(X_s) M(X)^{-1} M(X_s) \\ &= BB^T + N(X_s) N(X)^{-1} N(X_s) - [BB^T + N(X_s)][BB^T + N(X)]^{-1}[BB^T + N(X_s)]. \end{aligned} \tag{6.94}$$

Note that $Q(X, X_s)$ is a Schur complement of the block Ξ_{22} of the matrix Ξ defined as

$$\Xi = \begin{bmatrix} BB^T + N(X_s)N(X)^{-1}N(X_s) & BB^T + N(X_s) \\ BB^T + N(X_s) & BB^T + N(X) \end{bmatrix}. \tag{6.95}$$

Now, since (6.95) can be decomposed as

$$\Xi = \begin{bmatrix} B \\ B \end{bmatrix} [B^T \; B^T] + \begin{bmatrix} (N(X_s)N(X)^{-1} \\ I \end{bmatrix} N(X)[N(X)^{-1}N(X_s) \;\; I] \tag{6.96}$$

it follows from $\Xi \geq 0$ that $Q(X, X_s) \geq 0$, which implies that $T(X, X_s) \geq 0$, and so $DR(X) - R(X) \geq 0$ and thus $0 \geq DR(X) \geq R(X)$. □

6.3.3 Norm-bounding state-feedback control strategies

In this section, we discuss feedback controls that guarantee upper and lower bounds on the H_∞ norm of the system. The first Theorem presents the new lower bound on the optimum H_∞ norm achievable for a given system using state-feedback controls. For completeness, the statement of the Theorem includes the known upper bound on the H_∞ norm of a system.

Theorem 6.11. *If there exist $P_o \geq 0$ and a positive scalar γ_o such that*

$$P_o = H^T H + A^T P_o \left[I + \left(BB^T - \frac{1}{\gamma_0^2} GG^T \right) P_o \right]^{-1} A \tag{6.97a}$$

$$\gamma_o^2 I - G^T P_o G > 0 \tag{6.97b}$$

then the control $u = Kx$, with $K = -B^T P_o [I + (BB^T - \frac{1}{\gamma^2} GG^T) P_o]^{-1} A$ guarantees that

$$\lambda_{\max}(G^T P_o G) \leq \|T_c\|_\infty^2 < \gamma_o^2, \qquad T_c(z) = \begin{bmatrix} H \\ K \end{bmatrix} (zI - A - BK)^{-1} G \tag{6.97c}$$

and moreover

$$\lambda_{\max}(G^T P_o G) \leq \|T_{\min}\|_\infty^2 < \gamma_o^2. \tag{6.97d}$$

Proof:

a) The upper bounds are proved in Theorem 6.6, with the upper bound in (6.97d) following from the one in (6.97c) and the fact that $T_{\min}(z)$ has a lower H_∞-norm bound than that given by any other state-feedback control.

b) To show that the lower bounds hold, let $E = A + BK$ and $H_0 = \begin{bmatrix} H \\ K \end{bmatrix}$ with

$$K = -B^T P_o[I + (BB^T - \frac{1}{\gamma_o^2} GG^T)P_o]^{-1} A. \tag{6.98}$$

Then (E, H_o) is a detectable pair (Lemma 6.8). Define

$$\begin{aligned} \gamma_{so} &= \inf\{\gamma > 0 : \exists \text{ such that} P = H_o^T H_o + E^T P(I - \frac{1}{\gamma^2} GG^T P)^{-1} E, \gamma^2 I - G^T PG > 0\} \\ &= \|H_o(zI - E)^{-1} G\|_\infty = \|T_c\|_\infty \end{aligned}$$

and so the lower bound in (6.97b) is shown if we show that

$$\gamma_{so}^2 I \geq G^T P(\gamma_o) G. \tag{6.99}$$

Note that when $P = P_o$ and $\gamma = \gamma_o$ then (6.97a) can be rewritten as

$$P_o = H_o^T H_o + E^T P_o \left(I - \frac{1}{\gamma^2} GG^T P_o\right)^{-1} E \tag{6.100}$$

and since $\gamma_o^2 I - G^T P(\gamma_o) G > 0$, we have that $\gamma_o > \gamma_{\min}$ and also that $\gamma_o \geq \gamma_{so}$ by the definitions of $\gamma_{\min}$ and γ_{so}. Thus, by Theorem 6.7, we have $P(\gamma_{so}) \geq P(\gamma_o)$ and hence we have $\gamma_{so}^2 I \geq G^T P(\gamma_{so}) G \geq G^T P(\gamma_o) G$.

As for the second lower bound, we have, by Remark 6.2, that $P(\gamma_{\min}) \geq P(\gamma_0)$, and so by the definition of $\gamma_{\min}$, $\gamma_{\min}^2 I \geq G^T P(\gamma_{\min}) G \geq G^T P(\gamma_o) G$. □

We are now in a position to form an entire class of H_∞-norm-bounding state-feedback laws, for the system (6.50).

Theorem 6.12. *For $\gamma > \gamma_{\min}$, if $X \in S_\gamma$, then the control law $u_k = K_x x_k$, with $K_x = -B^T X \left[I + \left(BB^T - \frac{1}{\gamma^2} GG^T\right) X\right]^{-1} A$ guarantees that*

a) $A + BK_x$ is stable, and

b) $\lambda_{\max}^{1/2}(G^T X_s G) \leq \|T_c\|_\infty \leq \gamma$ where

$$T_c(z) = \begin{bmatrix} H \\ K \end{bmatrix} (zI - A - BK_x)^{-1} G.$$

Proof: By Lemma 6.12, $R(X) \leq 0$, and by Theorem 6.9, $\gamma^2 I - G^T X G > 0$. With $E = A + BK_x$ and H replaced with $\bar{H} = [H^T \; K_x^T]^T$ then by Lemma 6.7, $(E, \bar{H})$ is detectable and so Lemma 6.7 gives the upper bound.

The proof of the lower bound uses Theorem 6.8 since $X_s(\gamma)$, the stabilizing solution of $R(X) = 0$, with γ specified by in Theorem 6.12, satisfies $X \geq X_s(\gamma)$. With E and $\bar{H}$ so fixed, we consider the Riccati equation

$$P = \bar{H}^T \bar{H} + E^T P [I - UP]^{-1} E \tag{6.101}$$

and define

$$\gamma_1 = \inf\{(\gamma > 0 : P \text{ the stabilizing solution of (6.101) with } \gamma^2 I - G^T P G > 0\}.$$

Since $P_s(\gamma)$ is monotonically decreasing in γ, then $P_s(\gamma_1) \geq X_s(\gamma)$ and so $\gamma_1^2 I \geq G^T P_s(\gamma_1) G \geq G^T X_s(\gamma) G$, thus giving

$$\|T_c\|_\infty^2 = \gamma_1^2 \geq \lambda_{\max}\{(G^T X_s G\}. \qquad \Box$$

6.3.4 Computing H_∞-norm bounds for linear systems

Various methods have been proposed for finding either the H_∞ norm of a system, or bounds on its value. We review available lower and upper bounds and then, by appropriate modification of the result in the previous section, derive a new lower bound.

A straightforward means of computing a lower bound is simply by choosing $q_i \in [0, 2\rho]$, $i = 1, 2, \ldots, N_1$, and computing

$$\|T\|_\infty \geq \gamma_N \tag{6.102}$$

where

$$\gamma_N = \max_{1 \leq i \leq N_1} \lambda_{\max}^{1/2}\{T(e^{-j\theta})^T T(e^{j\theta})\}. \tag{6.103}$$

Using this method, we would expect to obtain a better approximation of the actual value as we increase N_1. Essentially, we have a search, but there is generally no clear means of obtaining a good estimate without computing a large number of values.

A simple lower bound on the H_∞ norm is the H_2 norm of the system, since the H_2 norm must be less than or equal to the H_∞ norm for all inputs. It can thus be shown that

$$\|T\|_2^2 = \text{ trace } (HL_cH^T) = \text{ trace } (G^TL_oG) \leq \|T\|_\infty \tag{6.104}$$

where L_c and L_o are the controllability and observability grammians

$$L_c = EL_cE^T + GG^T \tag{6.105a}$$

$$L_o = E^TL_oE + H^TH. \tag{6.105b}$$

This lower bound requires the solution of only one Lyapunov equation.

The Hankel (semi-) norm, which has proven to be useful in the approximation of high-order systems by low-order systems, may also be used. Given any LTI system we can consider the Hankel singular values σ_i, $i = 1, \ldots, n$, which may be shown to be

$$\sigma_i^2 = \lambda_i(L_oL_c), i = 1, \ldots, n. \tag{6.106}$$

It has been shown [75], [20], that

$$\|T\|_H = \sigma_{\max}(T) \leq \|T\|_\infty \leq 2\sum_{i=1}^{n}\sigma_i(T). \tag{6.107}$$

This gives a quick means of obtaining both upper and lower bounds on the H_∞ norm. The computation of these bounds requires the solution of two Lyapunov equations. It can be shown that $\|T\|_2 \leq \|T\|_H$, and so the Hankel singular values give a tighter lower bound than the H_2 norm.

The disadvantage of the last two bounds is that they are not always tight. In the discrete case we can establish a lower bound for $\|T\|_\infty$ and thus a interval within which the H_∞ norm of the system must lie. The following Lemma presents the new lower bound, together with the known upper bound on the H_∞ norm.

Lemma 6.13. *Let E be a stability matrix, and $T(z) = H(zI - E)^{-1}G$. If there exists $P \geq 0$ and a positive scalar γ such that*

$$P = H^TH + E^TP\left(I - \frac{1}{\gamma^2}GG^TP\right)^{-1}E \tag{6.108}$$

and

$$\gamma^2 I - G^T P G > 0. \tag{6.109}$$

Then

$$\lambda_{\max}(G^T P G) \leq \|T\|_\infty^2 < \gamma^2. \tag{6.110}$$

Proof. The upper bound is proved in Theorem 6.6. To show the lower bound, note that since

$$\begin{aligned}
\gamma_s &= \|H(zI - E)^{-1} G\|_\infty \\
&= \inf\{\gamma > 0 : \exists p \geq \text{ such that } P = H^T H + E^T P \left(I - \frac{1}{\gamma^2} G G^T P\right)^{-1} E \geq 0, \gamma^2 I - G^T P G > 0\}
\end{aligned}$$

then by Theorem 6.7 (which holds in particular for $B = 0$), $P(\gamma_s) \geq P(\gamma)$, and thus we have

$$\gamma_s^2 I \geq G^T P(\gamma_s) G \geq G^T P(\gamma_o) G. \qquad \square$$

We can now give an algorithm for computing H_∞ norm. This algorithm requires a given upper bound γ^u and a lower bound γ^l. We can set $\gamma^l = 0$ with γ^u a large number, or else select both bounds from (6.107).

Algorithm:

(1) Given E, G, H, γ^l, γ^u. Let $\gamma_1^u = \gamma^u$, and $\gamma_1^l = \gamma^l$, $\gamma_1 = \frac{1}{2}(\gamma^l + \gamma^u)$.

(2) Compute P_i (eqn. (6.108)), the stabilizing solution (if possible).

(3) If either:

a) no stabilizing solution exists or

b) $(\gamma_i^2 I - G^T P_i G)$ is not positive definite

then let $\gamma_{i+1}^l = \gamma_i$, and $\gamma_{i+1}^u = \gamma_i^u$; otherwise let $\gamma_{i+1}^l = \max\{\lambda_{\max}^{1/2}(G^T P_i G), \gamma_i^l\}$, and $\gamma_{i+1}^u = \gamma_i$.

(4) Let $\gamma_{i+1} = \frac{1}{2}(\gamma_{i+1}^l + \gamma_{i+1}^u)$.

(5) If $|\gamma_{i+1} - \gamma_i| \leq \epsilon$, a specified accuracy level then stop with $\|T\|_\infty \approx \gamma_{i+1}(\pm\epsilon)$ otherwise increment i and γ_o to (2). $\square$

6.3.5 Examples

In this section we consider two examples. The first is related to the computation of the optimal H_∞-norm-bounding state-feedback approach. The second relates to the computation of the H_∞ norm of a given system.

For the computation of the optimal state feedback consider the scalar system

$$\begin{aligned} x_{k+1} &= ax_k + bu_k + Gw_k \\ \zeta_k &= hx_k \end{aligned} \tag{6.111}$$

and also consider the DARE

$$P = h^2 + \frac{a^2 P}{1 + (b^2 - g^2/\gamma^2)P}. \tag{6.112}$$

Various phenomena can be illustrated with this system regarding the norm bounds discussed above. It is also possible to understand when the norm bounds are tight. The conditions necessary for obtaining the bounds are:

i) P is the stabilizing solution of (6.112), and

ii) $\gamma^2 - g^2 P > 0$ (the "convexity" condition).

Under these conditions, there exists a state feedback that guarantees an H_∞-norm bound less than γ. This example will demonstrate that as γ is decreased, that either condition, (i), or (ii) might be the first to fail, thus stopping the search for the minimizer.

Solving (6.112) for γ^2 gives

$$\gamma^2 = g^2 \frac{P(P - h^2)}{b^2P^2 - (b^2h^2 + a^2 - 1)P - h^2}. \tag{6.113}$$

Example 1: Let $a = b = g = h = 1/2$. Graphs of the relationship between P and γ^2 are shown in Figure 6.1. The figure shows both solutions to the Riccati equation when they exist. In this figure, we show a trial value γ_1^2, and the lower bound obtained from the computation of $P(\gamma_1)$, i.e., $g^2 P_1$.

For this system, the stabilizability condition defines the minimizer $P_{\min} = 1/2$, for which $\gamma^2 = 1/5$. This occurs at the local minimum of equation (6.113). For the corresponding

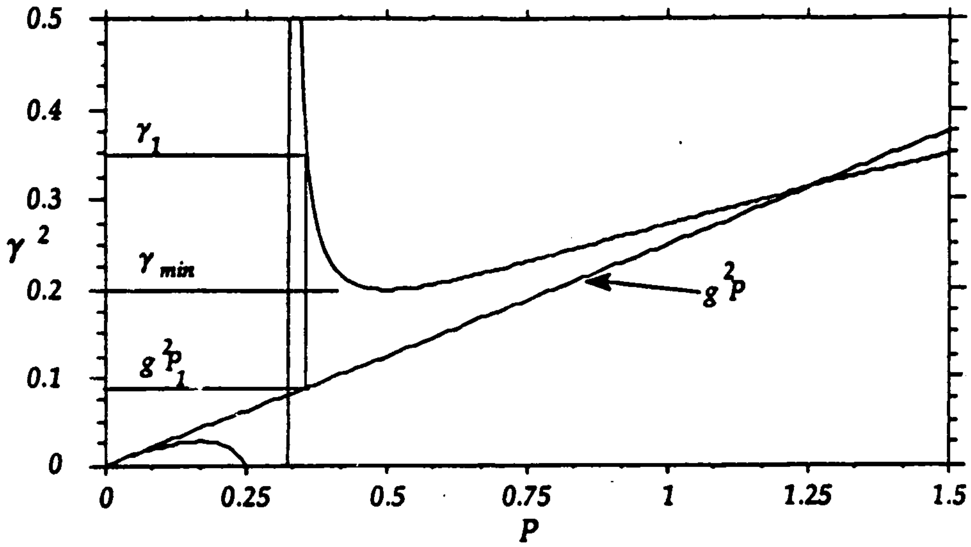

Figure 6.1: Unit circle eigenvalues constraining condition.

$K_{\min} = -1/4$, we have $E = A + BK_{\min} = 3/8$, stable. The H_∞ norm of the closed-loop system is

$$\left\| \begin{bmatrix} h \\ K_{\min} \end{bmatrix} (zI - E)^{-1} G \right\|_\infty^2 = \frac{(h^2 + K_{\min}^2) g^2}{(1 - |E|)^2} = \frac{1}{5} \tag{6.114}$$

Note also that $\gamma_2 P = (1/8)^2 < \|T_{\min}\|_\infty^2$ and so the lower bound is not tight for this system.

Example 2: $a = 1/2$, $b = g = h = 1$. A graph of γ^2 versus P is given in Figure 6.2. The state feedback that gives the minimal achievable H_∞ norm is $K_{\min} = -1/2$, which gives the H_∞ norm $\|T_{\min}\|_\infty^2 = 5/4$. Note that in this example the admissible minimum ($P = 2$), of equation (6.70) is on the boundary of the "convexity" region. Thus, the limiting condition here is the convexity condition. In this example, the lower bound is seen to be tight, since for $\gamma^2 = 5/4$, $P_s = 5/4 = \gamma_2 P_{\min}$, thus $\gamma_2 P_s = \|T_{\min}\|_\infty^2 = \gamma_2$. It can also be seen that at the minimum, $A_c = 2/5$, $K_{\min} = -1/2$, and $E = 0$.

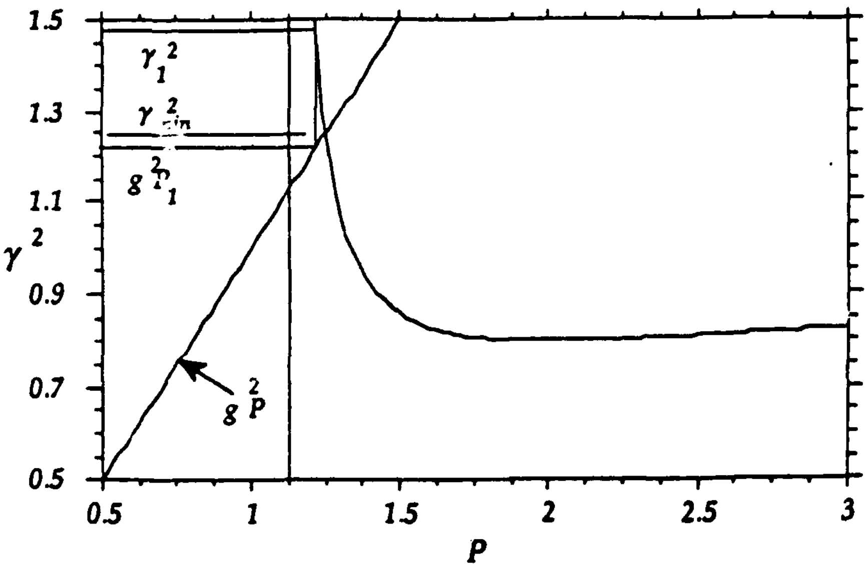

Figure 6.2: Convexity constraining condition.

Suppose that now we wish to compute the H_∞ norm of the stable system

$$\begin{aligned} x_{k+1} &= Ex_k + gw_k \\ \zeta_k &= Hx_k \end{aligned} \tag{6.115}$$

using the DARE

$$P = H^T H + \frac{E^2 P}{1 - g^2/\gamma^2 P} \tag{6.116}$$

which can be rewritten explicitly in terms of P:

$$\gamma^2 = g^2 \frac{P(P - H^T H)}{(1 - E^2)P - H^T H}. \tag{6.117}$$

Example 3: Figure 6.3 shows the graph of γ^2 versus P for the case of $E = 3/8$, $g = 1/2$, $H = [1/2 \ \ - \ \ 1/4]^T$.

Note that the lower bound $g^2 P$ is not tight. Indeed, it can be shown that for any γ, the difference between the lower bound (computed for γ) and the H_∞ norm satisfies

$$\|T\|_\infty^2 - L.B.(\gamma)^2 = \gamma_{\min}^2 - g^2 P(\gamma) \geq \frac{g^2 H^T H |E|}{(1 - |E|)2}. \tag{6.118}$$

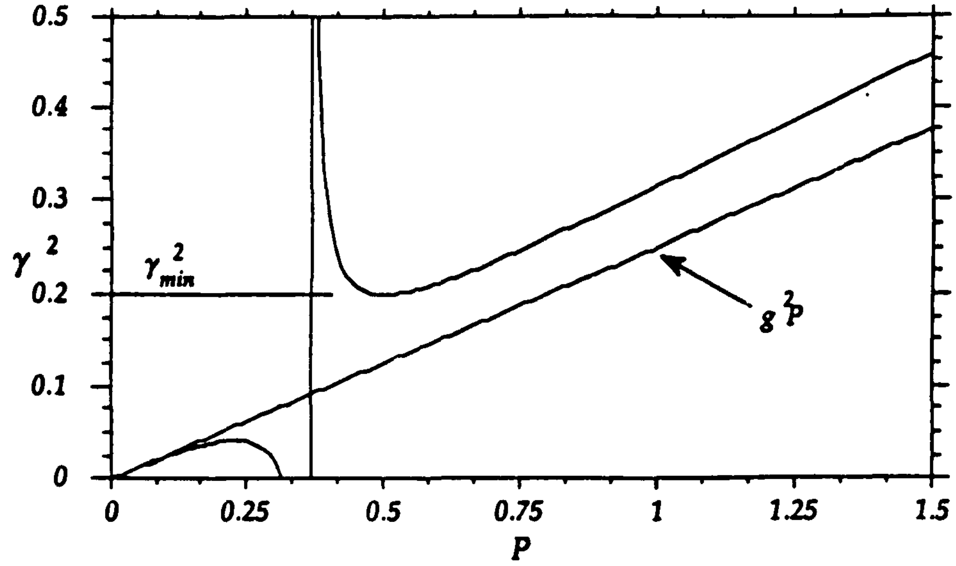

Figure 6.3: Computing the norm of a stable system.

Thus, we see that for the H_∞-optimal state feedback control problem, the convexity condition or the stabilizability condition could be active. In the case of the H_∞ norm calculation, however, the limiting case is always the stabilizability condition. These conclusions carry over to higher-order systems.

6.3.6 The observer-based H_∞-norm-bounding control

In this section we present the solution of the observer-based H_∞-norm-bounding problem when only certain output measurements are available to implement the control. The main results are stated in this subsection and proved in subsection 6.3.7. The results we present parallel those available for the continuous-time problem, but are established using a novel transformation of the the discrete algebraic Riccati equation (DARE).

Suppose that instead of having full state feedback, only certain (noisy) outputs are measured. We thus consider the system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Gw_k \\ y_k &= Cx_k + v_k \\ \zeta_k &= \begin{bmatrix} Hx_k \\ u_k \end{bmatrix} \end{aligned} \tag{6.119}$$

and assume that $v_k \in l_2$, and $[A, C]$ is detectable in addition to the assumptions made for the system (6.50). The problem is set up in a manner fully analogous to the continuous case. Thus, we adjoin to (6.71), the linear time-invariant controller

$$\xi_{k+1} = A\xi_k + B\hat{u}_k + G\hat{w}_k + L(y_k - Cx_k) \tag{6.120a}$$

$$\hat{u}_k = K\xi_k \tag{6.120b}$$

where K is given by (6.55) for given γ, and where $\hat{u}$ and $\hat{w}$ are estimates of the optimal control and "worst disturbance" from the state-feedback solution. The actual "worst disturbance", $\bar{w}$ is a disturbance that achieves the maximum input/output (I/O) energy ratio, i.e.,

$$\bar{w} = \arg \max_{w \in l_2 \ w \neq 0} \frac{||T_c(z)w(z)||_{l_2}}{||w(z)||_{l_2}} \tag{6.121}$$

where $T_c(z) = H_o(zI - E)^{-1}G$, $H_o = \begin{bmatrix} H \\ K \end{bmatrix}$, $E = A + BK$. It can be shown [40] that there exist "stochastic" disturbances in l_2, that achieves I/O energy ratios as high as that of $\bar{w}$, which can be shown to be realizable in feedback form,

$$\bar{w}_k = K_d x_k, \qquad K_d = \frac{1}{\gamma^2} G^T P \left[I + \left(BB^T - \frac{1}{\gamma^2} GG^T \right) P \right]^{-1} A \tag{6.122}$$

and where P is the stabilizing solution of (6.53).

Using the structure (6.120), we seek the matrix L to achieve stability and guarantee a desired H_∞-norm bound. To this end we form the closed loop system:

$$\begin{bmatrix} x \\ \xi \end{bmatrix}_{k+1} = \begin{bmatrix} A & BK \\ LC & A_c - LC \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}_k + \begin{bmatrix} G & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}_k \tag{6.123a}$$

with the output

$$\zeta_k = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}_k \tag{6.123b}$$

where $A_c = [I + (BB^T - \frac{1}{\gamma^2}GG^T)P]^{-1}A = A + BK + GK_d$. Let $e = x - \xi$, and introduce the transformation

$$\tilde{x} = \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \tag{6.124}$$

and apply it to (6.123) to obtain

$$\tilde{x}_{k+1} = \begin{bmatrix} E & -BK \\ -GK_d & A + GK_d - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}_k + \begin{bmatrix} G & 0 \\ G & -L \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}_k = E_o\tilde{x}_k + G_o\tilde{w}_k \tag{6.125}$$

and

$$\zeta_k = \begin{bmatrix} H & 0 \\ K & -K \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}_k = H_o\tilde{x}_k. \tag{6.126}$$

The central result of this section defines an observer-based controller that satisfies an uniform H_∞-norm bound, and can be stated as follows:

Theorem 6.13. *If there exists $P > 0$ satisfying*

$$P = H^TH + A^TP\left[I + \left(BB^T - \frac{1}{\gamma^2}GG^T\right)P\right]^{-1}A, \tag{6.127}$$

with $\gamma > 0$ and

$$\gamma^2 I - G^TPG > 0 \tag{6.128}$$

and if there exists $V > 0$ that satisfies

$$V = \frac{1}{\gamma^2}H^TH - C^TC + A^TV[I - GG^TV]^{-1}A, \tag{6.129}$$

with

$$\gamma^2 V > H^TH + A^TP\left[I - \frac{1}{\gamma^2}GG^TP\right]^{-1}A \tag{6.130}$$

and if the gain matrix L is chosen to be

$$L = \left(I - \frac{1}{\gamma^2}V^{-1}P\right)^{-1}A\left(V + C^TC - \frac{1}{\gamma^2}H^TH\right)^{-1}C^T \tag{6.131}$$

then for $T_c(z) = H_o(zI - E_o)^{-1}G_o$ we have

a) E_o is stable, and

b) $||T_c||_\infty < \gamma$.

Thus the solution of the problem is given in terms of two uncoupled Riccati equations, as in the continuous case.

A related result has been obtained by [70]. However, their conclusions are again based on the "perfect information" case. In addition, the solution of that problem is given in terms of two *coupled* Riccati equations.

The proof of Theorem 6.8 will be based on the relationship between the discrete algebraic Riccati equation (DARE) of dynamic games, and its relationship to a nonsymmetric, generalized, algebraic Riccati equation (GARE), and will be given in the next section. This novel approach also gives insight into the problem, using results in [76] regarding the solution of generalized Riccati equations.

6.3.7 The generalized Riccati equation

Recall the DARE

$$P = H^T H + A^T P \left[I + \left(BB^T - \frac{1}{\gamma^2} GG^T \right) P \right]^{-1} A \tag{6.132}$$

and the associated symplectic matrix

$$\$ = \begin{bmatrix} A + \left(BB^T - \frac{1}{\gamma^2} GG^T \right) A^{-T} H^T H & -\left(BB^T - \frac{1}{\gamma^2} GG^T \right) A^{-T} \\ -A^{-T} H^T H & A^{-T} \end{bmatrix}. \tag{6.133}$$

We now consider the GARE

$$X = (A^T X + H^T H) \left[A - \left(BB^T - \frac{1}{\gamma^2} GG^T \right) X \right] \tag{6.134}$$

or

$$-XA + A^{-T} \left[I + H^T H \left(BB^T - \frac{1}{\gamma^2} GG^T \right) A^{-T} \right] X - X \left(BB^T - \frac{1}{\gamma^2} GG^T \right) X - A^{-T} H^T H A = 0. \tag{6.135}$$

This is a continuous-time ARE, and has as its associated generator matrix

$$\not{c} = \begin{bmatrix} A & -\left(BB^T - \frac{1}{\gamma^2} GG^T \right) \\ -A^{-T} H^T H A & A^{-T} \left[I + H^T H \left(BB^T - \frac{1}{\gamma^2} GG^T \right) A^{-T} \right] \end{bmatrix}. \tag{6.136}$$

It can be shown [26] that $\$$ is a *symplectic* matrix since for $U = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, we have $U^{-1}\$^T U = \$^{-1}$. It can be shown that $¢$, on the other hand, is neither symplectic (since $U^{-1}¢^T U \neq ¢^{-1}$), nor *Hamiltonian* (since $U^{-1}¢^T U \neq -¢$) [26]. However, like $\$$, the matrix $¢$ has the property that if λ is an eigenvalue of $¢$, then $1/\lambda$ is also an eigenvalue. Indeed, we have:

Lemma 6.14. *$\$$ and $¢$ have the same spectrum.*

Proof. If two matrices T_1 and T_2 are square, then the spectrum of the product T_1T_2 is the same as the spectrum of T_2T_1. Examining $\$$ and $¢$, we have

$$\begin{aligned} \$ &= \begin{bmatrix} A & -\left(BB^T - \frac{1}{\gamma^2}GG^T\right) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A^{-T}H^TH & A^{-T} \end{bmatrix} = T_1T_2 \\ ¢ &= \begin{bmatrix} I & 0 \\ -A^{-T}H^TH & A^{-T} \end{bmatrix} \begin{bmatrix} A & -\left(BB^T - \frac{1}{\gamma^2}GG^T\right) \\ 0 & I \end{bmatrix} = T_2T_1. \end{aligned}$$

□

Remark 6.6: We take note from the proof of Lemma 6.14 that the invertibility of T_1 and T_2 depend only on the invertibility of A. Thus, if A is invertible, then neither $\$$ nor $¢$ have zero eigenvalues. Thus the "closed-loop" matrix

$$A_c = A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) A^{-T}(P - H^TH), \tag{6.137}$$

(which arises in Lemma 1.2), has no zero eigenvalues, and is thus invertible. □

Suppose the matrix Λ contains a subspectrum of $¢$. The next Lemma demonstrates the relation between the GARE and $¢$, based on [76].

Lemma 6.15. *If* $¢\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Lambda$, *then X solves (6.135).*

Proof. If

$$\begin{bmatrix} A & -\left(BB^T - \frac{1}{\gamma^2}GG^T\right) \\ -A^{-T}H^THA & A^{-T}\left[I + H^TH\left(BB^T - \frac{1}{\gamma^2}GG^T\right)A^{-T}\right] \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Lambda$$

then

$$A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right)X = \Lambda, \tag{6.138}$$

$$-A^{-T}H^THA + A^{-T}[I + H^TH\left(BB^T - \frac{1}{\gamma^2}GG^T\right)A^{-T}]X = X\Lambda, \tag{6.139}$$

and from (6.138), (6.139) follows

$$\begin{aligned}-H^THA + [I + H^TH\left(BB^T - \frac{1}{\gamma^2}GG^T\right)A^{-T}]X &= A^TX\Lambda, \\ &= A^TX\left[A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right)X\Lambda\right]\end{aligned}$$

or

$$(A^TX + H^TH)A - \left[I + (A^TX + H^TH)\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right]X = 0, \tag{6.140}$$

which reduces to

$$(A^TX + H^TH)\left[A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right)X\right] - X = 0.$$

$\square$

If $\rho(A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right)X) < 1$, (i.e., $\rho(\Lambda) < 1$ by (6.138)), then X is called the stabilizing solution of (6.135). Note that instead of the vectors $\begin{bmatrix} I \\ X \end{bmatrix}$, we may use the Schur vectors $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ if Λ is in real Schur form, or the eigenvectors $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ if Λ is in Jordan form, giving $X = X_2X_1^{-1}$ (or $X = Y_2Y_1^{-1}$) as a solution of (6.135). Also note that unlike the solution P, of the DARE (6.53), the solution X of the GARE is non-symmetric, while A^TX, however, is symmetric. We now demonstrate the relationship between the solutions of the GARE and the DARE.

Lemma 6.16. *If X and P are related by $A^TX = P - H^TH$ and*

$$\left[I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right]^{-1}$$

exists then P solves (6.53) iff X solves (6.135). Moreover, P is the stabilizing solution to (6.53) iff X is the stabilizing solution to (6.135).

Proof: By the Remark 6.5, we know that $\left[A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) A^{-T}(P - H^T H)\right]^{-1}$ exists. Also, by hypothesis, we have

$$X = A^{-T}(P - H^T H). \tag{6.141}$$

To prove the first assertion, note that X solves (6.135) if

$$(A^T X + H^T H)\left[A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) X\right] - X = 0.$$

By (6.141) the above expression is equivalent to

$$P\left[A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) A^{-T}(P - H^T H)\right] - A^{-T}(P - H^T H) = 0 \tag{6.142}$$

which is equivalent to

$$\left(I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right) A^{-T}(P - H^T H) = PA$$

or

$$A^{-T}(P - H^T H) = \left(I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right)^{-1} PA. \tag{6.143}$$

On the other hand, expression (6.142) is equivalent to

$$H^T H - P + A^T PA - A^T P\left(BB^T - \frac{1}{\gamma^2}GG^T\right) A^{-T}\left(P - H^T H\right) = 0. \tag{6.144}$$

In light of (6.143) and (6.144), expression (6.142) is equivalent to

$$\begin{aligned}
0 &= H^T H - P + A^T PA - A^T P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\left(I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right)^{-1} PA \\
&= H^T H - P + A^T\left\{I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right) - P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right\} \\
&\quad \left(I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right)^{-1} PA \\
&= H^T H - P + A^T\left(I + P\left(BB^T - \frac{1}{\gamma^2}GG^T\right)\right)^{-1} PA \\
&= H^T H - P + A^T P\left(I + \left(BB^T - \frac{1}{\gamma^2}GG^T\right) P\right)^{-1} A,
\end{aligned}$$

and so P solves (6.53) if X solves (6.135). For the proof of the second part, we have from the proof of Lemma 6.15 that X is the stabilizing solution of (6.135) if $A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) X$

is a stability matrix. But

$$A_c = A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) A^{-T}(P - H^TH) = A - \left(BB^T - \frac{1}{\gamma^2}GG^T\right) X,$$

which, by (6.137) is stable if P is the stabilizing solution of (6.53). □

Remark 6.7: When P^{-1} exists and P solves (6.53), the condition $P > H^TH$ is equivalent to $P^{-1} > \frac{1}{\gamma^2}GG^T - BB^T$. In Corollary 6.1, we note that the condition $\gamma^2 I - G^TPG > 0$, is equivalent to $P^{-1} > \frac{1}{\gamma^2}GG^T$, when P^{-1} exists, since $P^{1/2}\left(P^{-1} - \frac{1}{\gamma^2}GG^T\right)P^{1/2} > 0$ is equivalent to $I - \frac{1}{\gamma^2}G^TP^{1/2}P^{1/2}G > 0$, and implies that $P > H^TH$. Thus, in Corollary 6.1, when P^{-1} exists, $P > H^TH$ is equivalent to $\gamma^2 I - G^TPG > 0$.

Remark 6.8: Lemma 6.16 was proved for $R_1 = H^TH \geq 0$ and $R_2 = BB^T - \frac{1}{\gamma^2}GG^T$, which is sign-indefinite. The proof, however does not depend on the sign definiteness of R_1, and can be used for indefinite R_1. This fact will be exploited in the observer equations.

The motivation for using the GARE, instead of the DARE in the proof of Theorem 6.13 is that, by using the GARE, we arrive at expressions that, as will be shown, are similar to the corresponding continuous-time expressions, e.g., $K = -B^TX$, $K_d = \frac{1}{\gamma^2}G^TX$ [1], [13]. This similarity serves two purposes: First, it simplifies the expressions. Second, and perhaps more important, it provides insight from the results on the observer-based H_∞-norm-bounding problem in the continuous-time case, and provides guidelines for the solution of the discrete-time problem.

In many optimal control problems, duality plays an important role. In the search for a solution to the present problem, we again seek to exploit duality as was successfully accomplished in the continuous problem [1]. Thus, we consider an observer DARE,

$$Q = GG^T + AQ\left[I + \left(C^TC - \frac{1}{\gamma^2}H^TH\right)Q\right]^{-1}A^T. \tag{6.145}$$

which is the dual of the control DARE (6.132). We expect, by analogy, that the condition $\gamma^2 I - HQH^T > 0$, will be necessary; it implies that $V^{-1} = Q > GG^T$. Note that V satisfies

$$V = \frac{1}{\gamma^2}H^TH - C^TC + A^TV[I - GG^TV]^{-1}A, \tag{6.146}$$

which is the DARE found in Theorem 6.13. Along with V, by fully analogous development, we associate with the DARE (6.146) the GARE

$$W = \left(A^T W + \frac{1}{\gamma^2} H^T H - C^T C\right)(A + GG^T W), \tag{6.147}$$

where W is related to V by

$$A^T W = V - \frac{1}{\gamma^2} H^T H + C^T C. \tag{6.148}$$

We now return to the proof of Theorem 6.13.

Proof of Theorem 6.13: Define $\bar{P} = \begin{bmatrix} P & 0 \\ 0 & P_1 \end{bmatrix}$, where $P_1 = \gamma^2 V - P$. Using Corollary 6.1, we desire to show that with L given by (6.131), $\bar{P}$ is positive definite and satisfies

$$\bar{P} = H_o^T H_o + E_o^T \bar{P} \left(I - \frac{1}{\gamma^2} G_o^T G_o \bar{P}\right)^{-1} E_o \tag{6.149}$$

$$\gamma^2 I - G_o^T \bar{P} G_o > 0. \tag{6.150}$$

Hence, by Corollary 6.1, the closed-loop system is stable and its H_∞ norm is bounded by γ. Transforming the DARE to the associated GARE, leads to $\bar{X}$ satisfying

$$\bar{X} = (E_o^T \bar{X} + H_o^T H_o)\left(E_o + \frac{1}{\gamma^2} G_o G_o^T \bar{X}\right). \tag{6.151}$$

We now show that $\bar{X}$ of the form

$$\bar{X} = \begin{bmatrix} X & X_{12} \\ 0 & X_{22} \end{bmatrix}, \tag{6.152}$$

where X is the stabilizing solution of (6.135) while X_{12}, and X_{22} satisfy

$$E^T X_{12} = (K_d^T G^T X_{22} + K^T K), \tag{6.153}$$

$$X_{22} = \gamma^2 W - X_{12} - X, \tag{6.154}$$

solves the GARE (6.151). We then conclude, by the use of Lemma 6.16, that $\bar{P}$ is related to $\bar{X}$ via

$$E_o^T \bar{X} = \bar{P} - H_o^T H_o > H_o^T H_o, \tag{6.155}$$

also satisfies (6.150) and thus is the desired solution to the DARE (6.149).

Let

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = (E_o^T \bar{X} + H_o^T H_o)\left(E_o + \frac{1}{\gamma^2} G_o G_o^T \bar{X}\right) - \bar{X}, \tag{6.156}$$

and so (6.151) is established if $G_{11} = G_{12} = G_{21} = G_{22} = 0$. Note first that using (6.153), and the definitions of K_d and K, that

$$\begin{aligned} E_o^T \bar{X} + H_o^T H_o &= \begin{bmatrix} E^T & -K_d^T G^T \\ -K^T B^T & (A + GK_d - LC)^T \end{bmatrix} \begin{bmatrix} X & X_{12} \\ 0 & X_{22} \end{bmatrix} + \begin{bmatrix} H^T H + K^T K & -K^T K \\ -K^T K & K^T K \end{bmatrix} \\ &= \begin{bmatrix} E^T X + H^T H + K^T K & E^T X_{12} - K_d^T G^T X_{22} - K^T K \\ -KTB^T X - K^T K & (A + GK_d - LC)^T X_{22} - K^T B^T X_{12} - K^T K \end{bmatrix} \\ &= \begin{bmatrix} A^T X + H^T H & 0 \\ 0 & (A - LC)^T X_{22} + A^T X_{12} \end{bmatrix}. \end{aligned} \tag{6.157}$$

We also have

$$\begin{aligned} E_o + \frac{1}{\gamma^2} G_o G_o^T \bar{X} &= \begin{bmatrix} E & -BK \\ -GK_d & A + GK_d - LC \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} GG^T & GG^T \\ GG^T & GG^T + LLT \end{bmatrix} \begin{bmatrix} X & X_{12} \\ 0 & X_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_c & N \\ 0 & A_c + N + L\left(\frac{1}{\gamma^2} L^T X_{22} - C\right) \end{bmatrix} \end{aligned} \tag{6.158}$$

where

$$N = -BK + \frac{1}{\gamma^2} GG^T (X_{12} + X_{22}). \tag{6.159}$$

Thus, equation (6.156) becomes

$$\begin{aligned} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = &\begin{bmatrix} A^T X + H^T H & 0 \\ 0 & (A - LC)^T X_{22} + A^T X_{12} \end{bmatrix} \begin{bmatrix} A_c & N \\ 0 & A_c + N + L\left(\frac{1}{\gamma^2} L^T X_{22} - C\right) \end{bmatrix} \\ &- \begin{bmatrix} X & X_{12} \\ 0 & X_{22} \end{bmatrix}. \end{aligned} \tag{6.160}$$

This gives

$$\begin{aligned} G_{11} &= (A^T X + H^T H) A_c - X = (A^T X + H^T H)\left[A - \left(BB^T - \frac{1}{\gamma^2} GG^T\right) X\right] - X \\ &= 0, \end{aligned}$$

by (6.134) and Lemma 6.16. Next, note by inspection that $G_{21} = 0$. Introduce $M = BB^TX + \frac{1}{\gamma^2}GG^TX_{22}$, and observe that

$$\begin{aligned}
G_{12} &= (A^TX + H^TH)N - X_{12} \\
&= (A^TX + H^TH)\left[-BK + \frac{1}{\gamma^2}GG^T(X_{12} + X_{22})\right] - E^{-T}X^TM \\
&= (A^TX + H^TH)\left(M + \frac{1}{\gamma^2}GG^TE^{-T}X^TM\right) - E^{-T}X^TM \\
&= (A^TX + H^TH)\left(X^{-T}E^T + \frac{1}{\gamma^2}GG^T\right)E^{-T}X^TM - E^{-T}X^TM \\
&= \left\{(A^TX + H^TH)X^{-T}\left(E^T + \frac{1}{\gamma^2}X^TGG^T\right) - I\right\}E^{-T}X^TM \\
&= \{PX^{-T}A_c^T - I\}E^{-T}X^TM \\
&= \{PP^{-1} - I\}E^{-T}X^TM = 0.
\end{aligned}$$

Before proceeding to establish that $G_{22} = 0$, we first note from (6.153) and (6.154), that

$$X_{22} = \gamma^2W - X_{12} - X = \gamma^2W - X - E^{-T}(K_d^TG^TX_{22} + K^TK), \tag{6.161}$$

which implies that

$$(I + E^{-T}K_d^TG^T)X_{22} = \gamma^2W - X - E^{-T}K^TK, \tag{6.162}$$

and so

$$\begin{aligned}
A_c^TX_{22} &= (E^T + K_d^TG^T)X_{22} = E^T(I + E^{-T}K_d^TG^T)X_{22} \\
&= E^T(\gamma^2W - X - E^{-T}K^TK) = \gamma^2E^TW - A^TX + K^TK - K^TK \\
&= \gamma^2E^TA^{-T}(A^TW) - A^TX \qquad (6.163) \\
&= \gamma^2E^TA^{-T}\left(V - \frac{1}{\gamma^2}H^TH + C^TC\right) - P + H^TH.
\end{aligned}$$

From (6.163)

$$X_{22}^T = \left[\gamma^2E^TA^{-T}\left(V - \frac{1}{\gamma^2}H^TH + C^TC\right) - P + H^TH\right]^TA_c^{-1}$$

and so

$$X_{22} = A_c^{-T}\left[\gamma^2 E^T A^{-T}(V - \frac{1}{\gamma^2}H^T H + C^T C) - P + H^T H\right].$$

Now, using the definition of A_c from (6.137), and using (6.141) to relate X and P produces

$$X_{22} = P\left(P^{-1} - \frac{1}{\gamma^2}GG^T\right)A^{-T}(\gamma^2 V - H^T H + \gamma^2 C^T C) - A_c^{-T}A^T X$$

while from (6.146) and the facts that $X = PA_c$ and that $A^T X$ is symmetric we have

$$\begin{aligned}
X_{22} &= P\left[P^{-1} - \frac{1}{\gamma^2}GG^T\right]\left[(\gamma^2 V)^{-1} - \frac{1}{\gamma^2}GG^T\right]^{-1} A - PA \\
&= P\left\{P^{-1} - \frac{1}{\gamma^2}GG^T - (\gamma^2 V)^{-1} + \frac{1}{\gamma^2}GG^T\right\}\left[(\gamma^2 V)^{-1} - \frac{1}{\gamma^2}GG^T\right]^{-1} A
\end{aligned}$$

and using again (6.146) this finally produces

$$X_{22} = \left(I - \frac{1}{\gamma^2}PV^{-1}\right)A^{-T}\left(\gamma^2 V - H^T H + \gamma^2 C^T C\right). \tag{6.164}$$

Therefore (6.131) simplifies to

$$X_{22}^T L = \gamma^2 C^T. \tag{6.165}$$

Now consider G_{22}. Since $G_{11} = G_{12} = 0$, then $G_{22} = G_{22} + G_{11} + G_{12}$ and so we proceed to show that this sum is equal to zero. We have

$$\begin{aligned}
G_{22} &= G_{22} + G_{11} + G_{12} \\
&= [(A - LC)^T X_{22} + A^T X_{12}]\left[A_c + N + L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right)\right] - X_{22} \\
&\qquad +(A^T X + H^T H)A_c - X + (A^T X + H^T H)N - X_{12} \\
&= [(A - LC)^T X_{22} + A^T X_{12}]\left[A_c + N + L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right)\right] - (X + X_{12} + X_{22}) \\
&\qquad +(A^T X + H^T H)(A_c + N) \\
&= [(A - LC)^T X_{22} + A^T X_{12} + A^T X + H^T H]\left[A_c + N + L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right)\right] \\
&\qquad -(X + X_{11} + X_{22}) - (A^T X + H^T H)L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right) \\
&= [A^T(X + X_{12} + X_{22}) + H^T H - C^T L^T X_{22}]\left[A_c + N + L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right)\right] \\
&\qquad -(X + X_{11} + X_{22}) - (A^T X + H^T H)L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right)
\end{aligned}$$

and using (6.154) and (6.147), as well as equations (6.146), (6.157) and Remark 6.7,

$$\begin{aligned}
G_{22} &= [A^T(\gamma^2 W) + H^T H - \gamma^2 C^T C - C^T(L^T X_{22} - \gamma^2 C)] \\
&\qquad \times \left[A + GG^T W + L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right)\right] \\
&\qquad -(\gamma^2 W) - (A^T X + H^T H)L\left(\frac{1}{\gamma^2}L^T X_{22} - C\right) \\
&= \gamma^2\left\{\left[A^T W + \frac{1}{\gamma^2}H^T H - C^T C\right][A + GG^T W] - W\right\} = 0
\end{aligned}$$

and thus, (6.151) is satisfied. Now consider (6.155). We have, from (6.151)

$$\begin{aligned}
E_o^T \bar{X} + H_o^T H_o &= \begin{bmatrix} A^T X + H^T H & 0 \\ 0 & (A - LC)^T X_{22} + A^T X_{12} \end{bmatrix} \\
&= \begin{bmatrix} A^T X + H^T H & 0 \\ 0 & \gamma^2 A^T W - A^T X - \gamma^2 C^T C \end{bmatrix} \\
&= \begin{bmatrix} P & 0 \\ 0 & \gamma^2\left(V - \frac{1}{\gamma^2}H^T H + C^T C\right) - P + H^T H - \gamma^2 C^T C \end{bmatrix} \qquad (6.166) \\
&= \begin{bmatrix} P & 0 \\ 0 & \gamma^2 V - P \end{bmatrix} = \bar{P}.
\end{aligned}$$

Now $\bar{P}$ is positive definite if $P_1 = \gamma^2 V - P$ is positive definite, and this is established using (6.127), and (6.130):

$$\begin{aligned}
\gamma^2 V - P &> H^T H + A^T\left[P^{-1} - \frac{1}{\gamma^2}GG^T\right]^{-1} A - P \\
&\geq H^T H + A^T\left[P^{-1} + BB^T - \frac{1}{\gamma^2}GG^T\right]^{-1} A - P = 0;
\end{aligned} \qquad (6.167)$$

thus, (6.155) is established by (6.166) and (6.167). Finally, consider (6.150). We wish to show that conditions (6.128) and (6.130) imply

$$\gamma^2 I - G_o^T \bar{P} G_o > 0.$$

We recall from Remark 6.7, that (6.150) is equivalent to $\bar{P} > H_o^T H_o$, which we will demonstrate. Since P satisfies (6.127), then

$$(P - H^T H)^{-1} - A^{-1}BB^T A^{-T} = A^{-1}\left(P^{-1} - \frac{1}{\gamma^2}GG^T\right)A^{-T},$$

and so conditions (6.128) and (6.130) imply that

$$\gamma^2 V > H^T H + [(P - H^T H)^{-1} - A^{-1} B B^T A^{-T}]^{-1} > 0$$

or

$$(P_1 + P - H^T H) > [(P - H^T H)^{-1} - A^{-1} B B^T A^{-T}]^{-1} > 0 \qquad (6.168)$$

since $\gamma^2 V = P_1 + P$. Since $P > H^T H$, then (6.168) holds if

$$(P - H^T H)^{-1} - A^{-1} B B^T A^{-T} > (P_1 + P - H^T H)^{-1} > 0,$$

which is equivalent to

$$(P - H^T H)\{(P - H^T H)^{-1} - A^{-1} B B^T A^{-T} - (P_1 + P - H^T H)^{-1}\}(P - H^T H) > 0. \quad (6.169)$$

Since $K = -B^T A^{-T}(P - H^T H)$, using (6.169) we can write

$$\begin{aligned} K^T K &= (P - H^T H) A^{-1} B B^T A^{-T} (P - H^T H) \\ &< (P - H^T H)\{(P - H^T H)^{-1} - (P_1 + P - H^T H)^{-1}\}(P - H^T H) \\ &= [(P - H^T H)^{-1} + P_1^{-1}]^{-1}. \end{aligned}$$

This leads to

$$\begin{aligned} I &> K[(P - H^T H)^{-1} + P_1^{-1}]K^T \\ &= K[I \ -I] \begin{bmatrix} P - H^T H & 0 \\ 0 & P_1 \end{bmatrix}^{-1} \begin{bmatrix} I \\ -I \end{bmatrix} K^T = \tilde{K} \tilde{P}^{-1} \tilde{K}^T. \end{aligned}$$

Thus, conditions (6.128) and (6.130) imply that

$$\begin{aligned} 0 &< \tilde{P} - \tilde{K}^T \tilde{K} \\ &= \begin{bmatrix} P - H^T H & 0 \\ 0 & P_1 \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} K^T K [I \ -I] = \bar{P} - H_o^T H_o. \end{aligned}$$ □

Remark 6.9: The structures of L and K in this problem (i.e., $K = -B^T X$ and $L^T X_{22} = \gamma^2 C$) are similar to those of the continuous-time solutions, though they are not intuitive from the DARE equations. The use of the GARE framework thus provides an elegant solution to this problem. Numerically, the computation of the GARE itself should also be more robust, since no inversions are required.

Remark 6.10: If the worst disturbances are assumed, i.e.,

$$\begin{bmatrix} w \\ v \end{bmatrix} = \frac{1}{\gamma^2} G_o^T \bar{X} x,$$

then the total closed-loop matrix becomes

$$E_o + \frac{1}{\gamma^2} G_o G_o^T \bar{X} = \begin{bmatrix} A_c & N \\ 0 & A + GG^T W \end{bmatrix}. \tag{6.170}$$

Since $A + GG^T W$ is the "closed-loop" matrix associated with the observer Riccati equation, we see that the eigenvalues are the union of the control and observer closed-loop matrix eigenvalues.

Thus, we have established the observer-based controller analogous to that in the continuous-time case. The results here, because of the discrete Riccati equation structure, have added conditions (6.128) and (6.130). Note that (6.130) is a condition on the relationship between Riccati solutions (6.127) and (6.129). A similar condition is found in [48,1], Theorem 3].

6.3.8 A lower bound on the optimal H_∞ norm

As in the case of the state-feedback control for $\gamma > \gamma_{\min}$, it is possible to obtain both an upper and lower bound on the minimal achievable H_∞ norm when observer-based controls are used. Let γ^o be the the minimal achievable H_∞ norm using a controller of the form presented here, such that for all $\gamma > \gamma^o$, there exist P, V and L satisfying (6.127) - (6.131). The next theorem gives a lower bound on the value of γ^o.

Theorem 6.14. *If there exist positive-definite P and V such that*

$$\begin{aligned} P &= H^T H + A^T P \left[I + \left(BB^T - \frac{1}{\gamma^2} GG^T \right) P \right]^{-1} A \\ \gamma^2 V &= H^T H - \gamma^2 C^T C + \gamma^2 A^T V [I - GG^T V]^{-1} A \end{aligned}$$

with $\gamma > 0$, $\gamma^2 I - G^T P G > 0$, and $\gamma^2 V > H^T H + A^T P \left(I - \frac{1}{\gamma^2} GG^T P \right)^{-1} A$, then

$$\lambda_{\max}^{1/2} \left\{ \left[H^T H + A^T P \left(I - \frac{1}{\gamma^2} GG^T P \right)^{-1} A \right] V^{-1} \right\} \leq \gamma^o < \gamma. \tag{6.171}$$

The proof of the theorem requires two technical Lemmas relating the monotonicity of solutions of the DARE with respect to γ.

Lemma 6.17. $\left[P(\gamma)^{-1} - \frac{1}{\gamma^2}GG^T\right]^{-1}$ *is monotonically increasing in* γ *for* $\gamma \in (\gamma^o, 0)$.

Proof. By Theorem 6.7, $P(\gamma)$ is monotonically increasing for $\gamma \in (\gamma_{\min}, \infty)$. It can be easily shown that $\gamma^o \geq \gamma_{\min}$, and so $P(\gamma)$ is monotone in (γ^o, ∞). Thus, for $\gamma_1 > \gamma_2 > \gamma^o \geq \gamma_{\min}$, it follows that $P(\gamma_1) \leq P(\gamma_2) \leq P(\gamma^o,$, and so $P(\gamma_1)^{-1} \geq P(\gamma_2)^{-1} \geq P(\gamma^o)^{-1}$. We also have that $-\frac{1}{\gamma_1^2}GG^T \geq -\frac{1}{\gamma_2^2}GG^T \geq -\frac{1}{\gamma^{o2}}GG^T$, and thus we have

$$P(\gamma_1)^{-1} - \frac{1}{\gamma_1^2}GG^T \geq P(\gamma_2)^{-1} - \frac{1}{\gamma_2^2}GG^T \geq P(\gamma^o)^{-1} - \frac{1}{\gamma^{o2}}GG^T.$$

Therefore,

$$\left[P(\gamma_1)^{-1} - \frac{1}{\gamma_1^2}GG^T\right]^{-1} \leq \left[P(\gamma_2)^{-1} - \frac{1}{\gamma_2^2}GG^T\right]^{-1} \leq \left[P(\gamma^o)^{-1} - \frac{1}{\gamma^{o2}}GG^T\right]^{-1}. \qquad \square$$

Lemma 6.18. $V(\gamma)$ *is monotonically increasing in* γ *for* $\gamma \in (\gamma^o, \infty)$.

Proof. Recall that if Q is the stabilizing solution of (6.86a), by analogy with Theorem 6.7, Q is monotonically decreasing in γ for $\gamma \in (\bar{\gamma}_{\min}, \infty)$, where $\bar{\gamma}_{\min}$ is the smallest γ associated with the (dual) observer problem. It is easy to show that $\gamma^o \geq \bar{\gamma}_{\min}$. Thus, Q is monotonically decreasing in (γ^o, ∞), and so $V = Q^{-1}$ is monotone increasing there. $\square$

Proof of Theorem 6.14: The upper bound follows from the definition of γ^o. To prove the lower bound, we have to show that for $\epsilon > 0$

$$(\gamma^o + \epsilon)^2 V(\gamma) \geq H^T H + A^T \left(P(\gamma)^{-1} - \frac{1}{\gamma^2}GG^T\right)^{-1} A.$$

Without loss of generality, we can choose $\epsilon > 0$ such that $\gamma - \gamma^o > \epsilon > 0$. By Lemmas 6.17 and 6.18, we have that

$$\begin{aligned}
(\gamma^o + \epsilon)^2 V(\gamma) &\geq (\gamma^o + e)^2 V(\gamma^o + \epsilon) \\
&> H^T H + A^T \left[P(\gamma^o + \epsilon)^{-1} - \frac{1}{(\gamma^o + \epsilon)^2}GG^T\right]^{-1} A \\
&\geq H^T H + A^T \left(P(\gamma)^{-1} - \frac{1}{\gamma^2}GG^T\right)^{-1} A.
\end{aligned} \qquad \square$$

7 CONCLUSION

The research described in this report has provided many new results and research directions. These include the FH-norm-optimal low-order controller design problem with controllers restricted to belong to the class of projective controllers, the ARE-inequality based approach to designing H_∞-norm suboptimal full order controllers and its application to the decentralized control problem using full order controllers, the development of design equations for control systems reliable with respect to sensor and actuator outages, the design of strongly stable systems and decentralized control systems reliable with respect to a loss of certain control channels. In addition, initial results were obtained in the output-feedback-based H_∞-norm-bounding controls for discrete systems based on the transformation of the DARE to the GARE. These results include both an upper and a lower bound on the value of the H_∞ norm. Parametrizations of state-feedback and output-feedback controls that provide a specified bound on the H_∞ norm were also obtained.

The design of low-order controllers reduces ultimately to a nonlinear optimization problem, or to a problem of extracting elements of a subset to obtain a suboptimal solution. In this respect we have combined the projective controls approach with FH-norm optimization to simplify the computational aspects. The use of projective controls provides a convenient way of parametrizing the entire class of controllers of a given order that retain the dominant dynamics of a reference, state-feedback controlled system, which is computationally easy to design and adjust. Moreover, it provides a systematic procedure for retaining more reference dynamics by gradually increasing the controller order and the number of free design parameters. The FH norm provides a physically meaningful and computationally attractive criterion for optimal and suboptimal controller design. In particular, the possibility of transforming the parametrized class of projective controllers to a form linear in the free parameters has enabled the efficient application of FH-norm optimization to the optimization of the free design parameters in the parametrized class of projective controllers.

The described methodology for designing full order output feedback controllers with a guaranteed H_∞ norm bound highlight the possiblity of achieving additional important design goals with controllers of the same order as the plant. Considered here in detail is the problem of designing reliable controllers, and design equations have been developed which achieve reliability at the expense of an increase in the guaranteed bound on the H_∞-norm for both the base configuration and when certain outages of sensors or actuators occur. Results on the parametrization of classes of output feedback controllers that achieve an H_∞-norm bound we feel will allow a further and systematic development and evaluation of multiatribute designs. Finally, the DARE to GARE transformation and the resuilts already obtained concering the deeign equations for full order based output feedback control and the convexity properties of the discrete Riccati operator will, we expect, allow the results and insight gained from the analysis of the continuous control problems to be carried over to discrete control problem.

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